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ON TRANSFORMATIONS BY POLYNOMIALS IN TWO VARIABLES, II

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1. In [3] the following theorem was proved:

If $F_1(x,y)$ and $F_2(x,y)$ are forms over an algebraic number field K, which are without non-trivial common factor and are both of degree at least three, then the transformation T of the set K^2 defined by T: $(x,y)\to\big(F_1(x,y),F_2(x,y)\big)$ has no infinite invariant sets, and the same holds if both forms are of degree two.

In this note* we consider the case when the forms in question happen to have a common factor, and prove the following

THEOREM. *Let K be an imaginary quadratic extension of the rationals. Let $F_1(x,y)$ and $F_2(x,y)$ be forms over K with a common factor $F_0(x,y)$. Denote by n_i $(i=0,1,2)$ the degree of the form $F_i(x,y)$. If $2n_0+3 \leqslant \min(n_1,n_2)$, then the transformation T of K^2 defined by T: $(x,y) \to \big(F_1(x,y),F_2(x,y)\big)$ has no infinite invariant sets.*

2. LEMMA 1. *Suppose T is a transformation of a set Y onto itself and suppose that there exists a distinguished element a in Y such that $T(a)=a$ and, moreover, that there exists a function $f(y)$ defined in Y with values in the set of natural numbers, satisfying the following conditions:*

(i) *the equation $f(y)=c$ has for every natural c at most a finite number of solutions,*

(ii) *there exists a constant C such that from $f(y)\geqslant C$, $T(y)\neq a$ it follows $f\big(T(y)\big)>f(y)$.*

Then the set Y is finite.

Proof. Put $X=Y\setminus\{a\}$, $X_0=\{x\in X: T(x)\neq a\}$. Then evidently $T(X_0)=X$ and the conditions of lemma 1 in [2] are satisfied, hence X is finite and so must be Y, q. e. d.

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Every element ξ of K^2 can be uniquely represented in the form

$$\xi = \left\langle \frac{p_1^{(1)}+p_2^{(1)}\omega}{q_1}, \frac{p_1^{(2)}+p_2^{(2)}\omega}{q_2} \right\rangle,$$

where $1, \omega = (d+\sqrt{d})/2$ is an integral basis for K (d is the discriminant of K), $p_1^{(1)}, \ldots, p_2^{(2)}$ are rational integers, q_1, q_2 are natural numbers and $(p_1^{(i)}, p_2^{(i)}, q_i) = 1$ for $i = 1, 2$. For every such ξ put

$$f(\xi) = \max\{|p_1^{(1)}|, \ldots, |p_2^{(2)}|, q_1, q_2\}$$

and, moreover, for integral rational a, b put

$$\varphi(a+b\omega) = \max(|a|, |b|).$$

Let us remark that for x non-zero, integral in K we have $\|x\| \gg \varphi(x) \gg \|x\|$ (where $|a| \ll b$ means $a = O(b)$) and consequently for non-zero, integral x, y we have $\varphi(xy) \gg \varphi(x)\cdot\varphi(y)$.

LEMMA 2. *There exists a constant C such that from $f(\xi) \geqslant C$ and $T(\xi) \neq (0, 0)$ the inequality $f(T(\xi)) > f(\xi)$ follows.*

Proof. Suppose that for a sequence $\{\xi_k\}$ we have

(i) $f(\xi_k) \to \infty$,

(ii) $f(T(\xi_k)) \leqslant f(\xi_k)$,

(iii) $T(\xi_k) \neq (0, 0)$.

Let $F_1(x, y) = F_0(x, y)\bar{F}_1(x, y)$ and $F_2(x, y) = F_0(x, y)\bar{F}_2(x, y)$, where $\bar{F}_1(x, y)$ and $\bar{F}_2(x, y)$ are without any non-trivial common factor. We can assume that $\bar{F}_1(x, y)$ and $\bar{F}_2(x, y)$ have integral coefficients. Moreover, let Δ be the least positive rational integer such that $\bar{F}_0(x, y) = \Delta F_0(x, y)$ has integral coefficients.

Let $(q_1, q_2) = \varrho$ and $Q_i = q_i/\varrho$ $(i = 1, 2)$. (Here and in the sequel the indices k are omitted for the sake of simplicity.)

Now

$$T(\xi) = \left\langle \frac{\bar{F}_0^* \bar{F}_1^*}{\Delta(Q_1 Q_2 \varrho)^{n_1}}, \frac{\bar{F}_0^* \bar{F}_2^*}{\Delta(Q_1 Q_2 \varrho)^{n_2}} \right\rangle,$$

where $\bar{F}_i^* = \bar{F}_i\big(Q_2(p_1^{(1)}+p_2^{(1)}\omega), Q_1(p_1^{(2)}+p_2^{(2)}\omega)\big)$, $i = 0, 1, 2$. Let us denote by μ_i the greatest natural divisor of $\Delta(Q_1 Q_2 \varrho)^{n_i}$ which divides $\bar{F}_0^* \bar{F}_i$ $(i = 1, 2)$. Obviously

$$(1) \qquad f(T\xi) = \max\left\{\frac{\Delta(Q_1 Q_2 \varrho)^{n_1}}{\mu_1}, \frac{\Delta(Q_1 Q_2 \varrho)^{n_2}}{\mu_2}, \frac{\varphi(\bar{F}_0^* \bar{F}_1^*)}{\mu_1}, \frac{\varphi(\bar{F}_0^* \bar{F}_2^*)}{\mu_2}\right\}.$$

Consider at first the k's for which $\bar{F}_i^* = 0$ for some i and choose from them a subsequence with $i = 1$. (The remaining subsequence with $i = 2$ can be dealt with in exactly the same way. If any (or both) of the subsequences is finite, there is no need to consider it at all). In this case the ratio

$$\frac{p_1^{(1)}+p_2^{(1)}\omega}{Q_1} : \frac{p_1^{(2)}+p_2^{(2)}\omega}{Q_2}$$

can assume only a finite number of different values, say $\lambda_1, \ldots, \lambda_s$. (One of the λ's can be infinite if it happens that $p_1^{(2)} = p_2^{(2)} = 0$).

(a) $\bar{F}_1^* = 0, p_1^{(2)} = p_2^{(2)} = 0$. In this case obviously $q_2 = \varrho = 1$ and $\bar{F}_0^* \bar{F}_2^* = C_1(p_1^{(1)}+p_2^{(1)}\omega)^{n_2}$ with some integral C_1. By lemma 3 of [1] we have

$$\varphi(\bar{F}_0^* \bar{F}_2^*) \gg \max(|p_1^{(1)}|^{n_2}, |p_2^{(1)}|^{n_2}).$$

(This follows easily from $\varphi(x) \gg |x| \gg \varphi(x)$, but we prefer to quote [1] as the result there holds for every algebraic number field and we want to use the assumption that K is an imaginary quadratic field in those places only, where we are not able to avoid it).

Now consider μ_2 and remark that it must divide $\Delta Q_1^{n_2}$ and $C_1(p_1^{(1)} + p_2^{(1)}\omega)^{n_2}$, hence by the lemma 2 of [1] we get $\mu_2 \ll 1$.

It follows that

$$f(T(\xi)) \geqslant \max\left\{\frac{\varphi(\bar{F}_0^* \bar{F}_2^*)}{\mu_2}, \frac{\Delta Q_1^{n_2}}{\mu_2}\right\}$$

$$\gg \max\{Q_1^{n_2}, |p_1^{(1)}|^{n_2}, |p_2^{(1)}|^{n_2}\} = f^{n_2}(\xi),$$

thus $f(T(\xi)) > f(\xi)$ if $f(\xi)$ is sufficiently great.

(b) $\bar{F}_1^* = 0, |p_1^{(2)}|+|p_2^{(2)}| \neq 0$. In this case we can assume that

$$\frac{p_1^{(1)}+p_2^{(1)}\omega}{Q_1} = \lambda \cdot \frac{p_1^{(2)}+p_2^{(2)}\omega}{Q_2} \tag{2}$$

holds with a fixed λ.

Then

$$\bar{F}_0^* \bar{F}_2^* = \bar{F}_0(\lambda, 1)\bar{F}_2(\lambda, 1)Q_1^{n_2}(p_1^{(2)}+p_2^{(2)}\omega)^{n_2}$$

and, as

$$\varphi(Q_1^{n_2}(p_1^{(2)}+p_2^{(2)}\omega)^{n_2}) \gg \max(|Q_1 p_1^{(2)}|^{n_2}, |Q_1 p_2^{(2)}|^{n_2})$$

and $\bar{F}_0(\lambda, 1)\bar{F}_2(\lambda, 1)$ is fixed and non-zero, it results

$$\varphi(F_0^* F_2^*) \gg Q_1^{n_2}\max(|p_1^{(2)}|^{n_2}, |p_2^{(2)}|^{n_2}). \tag{3}$$

Consider now μ_2. It divides $\Delta(Q_1 Q_2 \varrho)^{n_2}$ and $\bar{F}_0^* \bar{F}_2^*$. The number $\bar{F}_0(\lambda, 1)\bar{F}_2(\lambda, 1)$ need not to be integral in K but in any case μ_2 divides $C_2(p_1^{(2)}Q_1+p_2^{(2)}Q_1\omega)^{n_2}$ with some C_2 which is integral in K. Let us denote by ν the greatest natural divisor of $\Delta(Q_1 Q_2 \varrho)^{n_2}$ dividing

$C_2(p_1^{(2)}+p_2^{(2)}\omega)^{n_2}Q_1^{n_2}$. Evidently $Q_1^{n_2}$ divides ν. Let $\nu' = \nu/Q_1^{n_2}$. It must be of the form $\nu' = AB$, where A divides Δ and B divides $q_2^{n_2}$. As no integral rational divisor ($\neq \pm 1$) of B divides $p_1^{(2)}+p_2^{(2)}\omega$, the application of the lemma 2 of [1] leads us to $B \ll 1$, and $\nu' \ll 1$ follows immediately. We have thus

$$\nu = C_3 Q_1^{n_2} \tag{4}$$

with bounded C_3.

Let us now return to (2). If

$$\lambda = \frac{1}{\gamma}(\alpha+\beta\omega),$$

then

$$\frac{p_1^{(1)}+p_2^{(1)}\omega}{Q_1} = \frac{r_1+r_2\omega}{\gamma Q_2}$$

with some rational integral r_1, r_2 and so Q_1 divides γQ_2. But γ is fixed and $(Q_1, Q_2) = 1$, hence it follows $Q_1 \ll 1$. From (4) we infer now $\nu \ll 1$ and a fortiori $\mu_2 \ll 1$.

From (1) and (3) it follows that

$$f\big(T(\xi)\big) \gg \max\{q_2^{n_2}, \varphi(\overline{F}_0^*\overline{F}_2^*)\} \gg \max\{q_2^{n_2}, |p_1^{(2)}|^{n_2}, |p_2^{(2)}|^{n_2}\}. \tag{5}$$

If $f(\xi) = q_2$, $|p_1^{(2)}|$ or $|p_2^{(2)}|$, then we get the desired result in the same way as in the previous case. The same method applies moreover if $f(\xi)$ is $o(q_2^{n_2})$ or $o(|p_i^{(2)}|^{n_2}$ with some $i = 1, 2$. Hence we can assume that

$$|p_1^{(2)}|^{n_2} \ll |p_{j_0}^{(1)}| = f(\xi), \quad |p_2^{(2)}|^{n_2} \ll |p_{j_0}^{(1)}| = f(\xi). \tag{6}$$

But $Q_2(p_1^{(1)}+p_2^{(1)}\omega) = \lambda Q_1(p_1^{(2)}+p_2^{(2)}\omega)$ and by taking norms of both sides we get

$$Q_2^2 N(p_1^{(1)}+p_2^{(1)}\omega) = N(\lambda)Q_1^2 N(p_1^{(2)}+p_2^{(2)}\omega). \tag{7}$$

As $N(x+y\omega) \gg \max\{|x|^2, |y|^2\}$ the left-hand side of (7) is $\gg |p_{j_0}^{(1)}|^2$, but the right-hand side is $\ll \max\limits_i(|p_1^{(2)}|^2) \ll |p_{j_0}^{(1)}|$ by (6). Thus we have got a contradiction.

It remains to consider the case of the k's for which $F_1^*F_2^* \neq 0$.

At first we consider the k's for which $f(\xi) = \max(q_1, q_2)$. Let $\max(Q_1, Q_2) = Q^*$, $\min(Q_1, Q_2) = Q_*$, $\max(n_1, n_2) = M$, $\min(n_1, n_2) = N$. From (1) and $f\big(T(\xi)\big) \leqslant f(\xi)$ we get

$$\mu_i \gg Q_*^{n_i}(\varrho Q^*)^{n_i-1} \quad (i = 1, 2). \tag{8}$$

As $\overline{F}_1(x, y)$ and $\overline{F}_2(x, y)$ have no non-trivial factor in common, it follows in the same way as in [3] that the largest natural divisor of (μ_1, μ_2) dividing $\overline{F}_1^*$ and $\overline{F}_2^*$ must be bounded, whence $(\mu_1, \mu_2) \ll |\overline{F}_0^*|$. Moreover, $[\mu_1, \mu_2] \ll (Q_1 Q_2 \varrho)^M$ and so

$$\mu_1\mu_2 = [\mu_1, \mu_2](\mu_1, \mu_2) \ll (\varrho Q_1 Q_2)^M |\overline{F}_0^*| \ll \varrho^{M+n_0} Q_*^M (Q^*)^{M+2n_0},$$

as $\overline{F}_0^* \ll \varrho^{n_0}(Q^*)^{2n_0}$. But (8) implies

$$\mu_1\mu_2 \gg Q_*^{M+N} \varrho^{M+N-1} (Q^*)^{M+N-1}$$

and so we must have

$$\varrho^{N-n_0-1} Q_*^N (Q^*)^{N-2n_0-1} \ll 1.$$

Now by assumption $N \geqslant 2n_0+3$ we have $f(\xi) = \varrho Q^* \ll 1$, a contradiction with (i).

Finally consider the case of the k's for which $f(\xi) = \max(|p_i^{(j)}|)$. Choose from them a subsequence with $f(\xi) = |p_1^{(2)}|$. (The remaining subsequences, if infinite, can be dealt with in the same way).

From (1) and $f\big(T(\xi)\big) \leqslant f(\xi)$ relations

$$(9) \qquad \mu_i \gg \frac{(\varrho Q_1 Q_2)^{n_i}}{|p_1^{(2)}|}, \qquad \mu_i \gg \frac{\varphi(\overline{F}_0^* \overline{F}_i^*)}{|p_1^{(2)}|} \qquad (i = 1, 2)$$

follow and, similarly as above, we get

$$\mu_1\mu_2 \ll |\overline{F}_0^*| (\varrho Q_1 Q_2)^M.$$

From (9) we infer that

$$\mu_1\mu_2 \gg \frac{(\varrho Q_1 Q_2)^{n_1} \varphi(\overline{F}_0^* \overline{F}_2^*)}{|p_1^{(2)}|^2}$$

and

$$\mu_1\mu_2 \gg \frac{(\varrho Q_1 Q_2)^{n_2} \varphi(\overline{F}_0^* \overline{F}_1^*)}{|p_1^{(2)}|^2}.$$

Consequently

$$|p_1^{(2)}|^2 |\overline{F}_0^*| \gg (\varrho Q_1 Q_2)^{n_1 - M} \varphi(\overline{F}_0^* \overline{F}_2^*)$$

and

$$|p_1^{(2)}|^2 |\overline{F}_0^*| \gg (\varrho Q_1 Q_2)^{n_2 - M} \varphi(\overline{F}_0^* \overline{F}_1^*).$$

By applying $\varphi(xy) \gg \varphi(x) \cdot \varphi(y)$ we obtain

$$|p_1^{(2)}|^2 \gg (\varrho Q_1 Q_2)^{n_1 - M} \varphi(\overline{F}_2^*), \qquad |p_1^{(2)}|^2 \gg (\varrho Q_1 Q_2)^{n_2 - M} \varphi(\overline{F}_1^*)$$

and from $\varphi(x) \gg |x|$ it follows that

$$|p_1^{(2)}|^2 \gg (\varrho Q_1 Q_2)^{n_1-M}|\bar{F}_2^*|, \qquad |p_1^{(2)}|^2 \gg (\varrho Q_1 Q_2)^{n_2-M}|\bar{F}_1^*|.$$

But

$$\bar{F}_i^* = Q_1^{n_i}(p_1^{(2)}+p_2^{(2)}\omega)^{n_i}\bar{F}_i\left(\frac{Q_2(p_1^{(1)}+p_2^{(1)}\omega)}{Q_1(p_1^{(2)}+p_2^{(2)}\omega)}, 1\right),$$

hence

$$|p_1^{(2)}|^2 \gg \varrho^{n_1-M}Q_1^{n_1+n_2-M}Q_2^{n_1-M}|p_1^{(2)}+p_2^{(2)}\omega|^{n_2}\left|\bar{F}_2\left(\frac{Q_2(p_1^{(1)}+p_2^{(1)}\omega)}{Q_1(p_1^{(2)}+p_2^{(2)}\omega)}, 1\right)\right|,$$

$$|p_1^{(2)}|^2 \gg \varrho^{n_2-M}Q_1^{n_1+n_2-M}Q_2^{n_2-M}|p_1^{(2)}+p_2^{(2)}\omega|^{n_1}\left|\bar{F}_1\left(\frac{Q_2(p_1^{(1)}+p_2^{(1)}\omega)}{Q_1(p_1^{(2)}+p_2^{(2)}\omega)}, 1\right)\right|$$

and so it follows that

$$\left|\bar{F}_1\left(\frac{Q_2(p_1^{(1)}+p_2^{(1)}\omega)}{Q_1(p_1^{(2)}+p_2^{(2)}\omega)}, 1\right)\right| + \left|\bar{F}_2\left(\frac{Q_2(p_1^{(1)}+p_2^{(1)}\omega)}{Q_1(p_1^{(2)}+p_2^{(2)}\omega)}, 1\right)\right| \ll \frac{1}{|p_1^{(2)}|} \to 0,$$

which contradicts $|\bar{F}_1|+|\bar{F}_2| > \delta > 0$. The lemma is thus proved in all cases, and the theorem follows immediately.

REFERENCES

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