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A CHARACTERIZATION OF THE DISC D^n

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It is well known that the sphere S^n is characterized by the existence of a Morse function with two critical points. The aim of this note is to prove that the disc D^n is characterized by the existence of an m-function with one critical point.

1. Definitions and notation. Let $(M^n, \partial M^n)$ be a smooth, compact, n-dimensional manifold with boundary ∂M^n. A smooth function $h\colon M^n \to R$ is called an *m-function* [1] if it has the following properties:

(a) h is a Morse function on M^n,

(b) h has no critical points on ∂M^n,

(c) $h|\partial M^n\colon \partial M^n \to R$ is a Morse function on ∂M^n.

m-functions exist on every compact manifold [1].

If $p \in M^n - \partial M^n$ is a critical point of h, its index is defined as that in the Morse theory, and if $x \in \partial M^n$ is a critical point of $h|\partial M^n$, its index is an ordered pair (λ, ε), where λ is the index of $h|\partial M^n$ and

$$\varepsilon = \begin{cases} +1 & \text{if } \operatorname{grad}_x h \text{ points outwards}, \\ -1 & \text{if } \operatorname{grad}_x h \text{ points inwards}. \end{cases}$$

For an m-function h, denote by $C_\lambda(h)$, $D_\lambda^+(h)$ and $D_\lambda^-(h)$ the numbers of critical points of indices λ, $(\lambda, +1)$ and $(\lambda, -1)$, respectively. Put

$$\mu^*(M^n, \partial M^n) = \inf\Big\{\sum_{i=0}^{n} C_i(h) + \sum_{i=0}^{n-1} D_i^+(h)\colon h \text{ is an } m\text{-function on } M^n\Big\}.$$

2. Lemmas.

LEMMA 1. *If on the manifold M^n there is an m-function h such that $D_i^+(h) = 0$ for each $i = 0, 1, \ldots, n-1$, then on the triad $(M^n, \partial M^n, \emptyset)$ there is a Morse function with $\sum_{i=1}^{n} C_i(h)$ critical points.*

Proof. With no loss of generality, we can assume $h\colon M^n \to (0, +\infty)$. Let $\varphi\colon \partial M^n \times [0, +\infty) \to M^n$ be a collar neighborhood of the boundary ∂M^n [3]. We take a Riemann metric on M^n such that $\boldsymbol{n} = -\partial/\partial t$

is the outward vector field normal to ∂M^n. This implies that the angle between vector fields $\boldsymbol{n}$ and $\operatorname{grad} h$ is not zero on ∂M^n. By the compactness of M^n, there is a real number $\varepsilon > 0$ such that the angle between $\boldsymbol{n}$ and $\operatorname{grad} h$ is not zero on $\varphi(\partial M^n \times [0, \varepsilon])$. Therefore, for every pair (x_0, t_0) in $\partial M^n \times [0, \varepsilon]$,

(1) if

$$\left.\frac{\partial h \circ \varphi(x,t)}{\partial x_i}\right|_{(x_0,\, t_0)} = 0 \quad \text{for each } i = 1, 2, \dots, n-1,$$

then

$$\left.\frac{\partial h \circ \varphi(x,t)}{\partial t}\right|_{(x_0,\, t_0)} > 0.$$

Let a: $[0, +\infty) \to [0, +\infty)$ be a smooth function with the following properties: $a(t) = 1$ for $t \geqslant \varepsilon$, $a(0) = 0$, and $da(t)/dt > 0$ for $0 \leqslant t < \varepsilon$.

Consider a function f: $M^n \to R$ defined by

$$f(p) = \begin{cases} h(p) & \text{if } p \notin \varphi(\partial M^n \times [0, \varepsilon]), \\ h(p)\,a(t) & \text{if } p = \varphi(x, t) \in \varphi(\partial M^n \times [0, \varepsilon]). \end{cases}$$

We show that f is a Morse function on the triad $(M^n, \partial M^n, \emptyset)$. Obviously, f is a smooth function on M^n and $f^{-1}(0) = \partial M^n$. Consider critical points of f on $\varphi(\partial M^n \times [0, \varepsilon])$. Since, by the definition of f,

$$\left.\frac{\partial f \circ \varphi(x,t)}{\partial x_i}\right|_{(x_0,\, t_0)} = 0 \quad \text{for each } i = 1, 2, \dots, n-1$$

and

$$\left.\frac{\partial f \circ \varphi(x,t)}{\partial t}\right|_{(x_0,\, t_0)} = 0,$$

we get

(2)
$$a(t_0)\left.\frac{\partial h \circ \varphi(x,t)}{\partial x_i}\right|_{(x_0,\, t_0)} = 0 \quad \text{for each } i = 1, 2, \dots, n-1,$$
$$a(t_0)\left.\frac{\partial h \circ \varphi(x,t)}{\partial t}\right|_{(x_0,\, t_0)} + h \circ \varphi(x_0, t_0)\left.\frac{da(t)}{dt}\right|_{t=t_0} = 0.$$

Consider two cases.

Case I. $t_0 > 0$. In this case

$$a(t_0) > 0 \quad \text{and} \quad \left.\frac{\partial h \circ \varphi(x,t)}{\partial x_i}\right|_{(x_0,\, t_0)} = 0 \text{ for each } i = 1, 2, \dots, n-1,$$

whence, by (1),

$$\left.\frac{\partial h\circ\varphi(x,t)}{\partial t}\right|_{(x_0,\,t_0)} > 0.$$

Since, moreover,

$$a(t_0) > 0,\qquad h\circ\varphi(x_0,t_0) > 0\quad\text{and}\quad \left.\frac{da(t)}{dt}\right|_{t=t_0} > 0,$$

condition (2) is not satisfied. Therefore, case I is impossible.

Case II. $t_0 = 0$. Here

$$a(t_0) = 0\quad\text{and}\quad \left.\frac{da(t)}{dt}\right|_{t=t_0} > 0,$$

whence

$$h\circ\varphi(x_0,t_0)\left.\frac{da(t)}{dt}\right|_{t=t_0} > 0,$$

a contradiction. Thus we have proved that f has no critical points on $\varphi(\partial M^n\times[0,\varepsilon])$. Since $f = h$ on $M^n-\varphi(\partial M^n\times[0,\varepsilon])$, f is a Morse function with $\sum_{i=0}^{n} C_i(h)$ critical points.

LEMMA 2. *If on the manifold M^n there is an m-function h such that $D_{n-1}^{+}(h)\geqslant 1$, then there is also an m-function $\tilde{h}$ such that*

$$C_i(\tilde{h}) = C_i(h)\ \textit{for}\ i = 0,1,\ldots,n-1\quad\textit{and}\quad C_n(\tilde{h}) = C_n(h)+1,$$
$$D_i^{+}(\tilde{h}) = D_i^{+}(h)\ \textit{for}\ i = 0,1,\ldots,n-2\quad\textit{and}\quad D_{n-1}^{+}(\tilde{h}) = D_{n-1}^{+}(h)-1,$$
$$D_i^{-}(\tilde{h}) = D_i^{-}(h)\ \textit{for}\ i = 0,1,\ldots,n-2\quad\textit{and}\quad D_{n-1}^{-}(\tilde{h}) = D_{n-1}^{-}(h)+1.$$

Proof. Consider a point $p\in\partial M^n$ with index $(n-1,\,+1)$. Then there are [1] a real number $0<\varepsilon<1/2$, a neighborhood U of p, and a coordinate system $\varphi\colon U\to R^n$ such that

$$\varphi(U) = \{(x_1,\ldots,x_n)\in R^n\colon x_n\leqslant -x_1^2-\ldots-x_{n-1}^2\ \text{and}\ x_1^2+\ldots+x_n^2<2\varepsilon\},$$
$$\varphi(U\cap\partial M^n) = \{(x_1,\ldots,x_n)\in\varphi(U)\colon x_n = -x_1^2-\ldots-x_{n-1}^2\},$$
$$h\circ\varphi^{-1}(x_1,\ldots,x_n) = x_n.$$

Let $A>e$ be a real number. Consider two functions $\psi,\eta\colon R\to R$ defined by

$$\psi(t) = \begin{cases} e^{-\varepsilon/t} & \text{if } t>0,\\ 0 & \text{if } t\leqslant 0\end{cases}\qquad\text{and}\qquad \eta(t) = At.$$

On $\varphi(U)$ we define a function $\tilde{f}$ by

$$\tilde{f}(x_1, \ldots, x_n) = x_n + \psi(x_n - x_1^2 - \ldots - x_{n-1}^2 + \varepsilon) \cdot \eta(-x_1^2 - \ldots - x_{n-1}^2 - x_n).$$

The function $\tilde{f}$ has the following properties: if $x_n = -x_1^2 - \ldots - x_{n-1}^2$ or if $x_1, \ldots, x_n \in \varphi(U)$ and $x_1^2 + \ldots + x_n^2 > \varepsilon$, then

$$\tilde{f}(x_1, \ldots, x_n) = h \circ \varphi^{-1}(x_1, \ldots, x_n);$$

$$\frac{\partial \tilde{f}}{\partial x_n}(0, \ldots, 0) < 0.$$

The function $\tilde{f}$ has only one critical point on $\varphi(U)$. This point has coordinates $(0, \ldots, 0, \zeta)$, where $-\varepsilon/2 < \zeta < 0$, it is non-degenerated and of index n.

Consider a function $\tilde{h}\colon M^n \to R$ defined by

$$\tilde{h}(p) = \begin{cases} h(p) & \text{if } p \notin U, \\ \tilde{f} \circ \varphi(p) & \text{if } p \in U. \end{cases}$$

It is clear that $\tilde{h}$ has the desired properties.

3. Main result.

THEOREM. *For every compact manifold $(M^n, \partial M^n)$ with boundary ∂M^n, the inequality $\mu^*(M^n, \partial M^n) \geqslant 1$ holds. The equality $\mu^*(M^n, \partial M^n) = 1$ occurs if and only if $(M^n, \partial M^n)$ is the pair (D^n, S^{n-1}).*

Proof. Let $h\colon M^n \to R$ be an m-function and let x_0 be a point such that

$$h(x_0) = \sup_{x \in M^n} h(x).$$

If $x_0 \in M^n - \partial M^n$, then $C_n(h) \geqslant 1$, and if $x_0 \in \partial M^n$, then $D_{n-1}^+(h) \geqslant 1$. Therefore,

$$\sum_{i=0}^{n} C_i(h) + \sum_{i=0}^{n-1} D_i^+(h) \geqslant 1,$$

and this implies $\mu^*(M^n, \partial M^n) \geqslant 1$.

To prove the second part, recall [1] that on D^n there exists an m-function h satisfying

$$\sum_{i=0}^{n} C_i(h) + \sum_{i=0}^{n-1} D_i^+(h) = 1.$$

Conversely, if $\mu^*(M^n, \partial M^n) = 1$, then on M^n there exists an m-function h such that

$$\sum_{i=0}^{n} C_i(h) + \sum_{i=0}^{n-1} D_i^+(h) = 1.$$

Two cases are possible.

Case I. $C_i(h) = 0$ and $D_i^+(h) = 0$ for $i = 0, 1, \ldots, n-1$, and $C_n(h) = 1$.

Case II. $C_i(h) = 0$ for $i = 0, 1, \ldots, n$, $D_i^+(h) = 0$ for $i = 0, 1, \ldots$ $\ldots, n-2$, and $D_{n-1}^+(h) = 1$.

In the first case, on the triad $(M^n, \partial M^n, \emptyset)$ there exists, by Lemma 1, a Morse function $f\colon M^n \to R$ which has only one critical point. This implies $(M^n, \partial M^n) = (D^n, S^{n-1})$ (see [2]).

In the second case we apply first Lemma 2 and thus get an m-function h on M^n as considered in Case I, which implies $(M^n, \partial M^n) = (D^n, S^{n-1})$.

REFERENCES

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