COLLOQUIUM MATHEMATICUM

VOL. LV 1988 FASC. 2

GROWTH OF THE NORMS OF PRODUCTS OF RANDOMLY DILATED FUNCTIONS FROM $A(T)$

BY

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1. Let $A(T)$ be the Banach algebra of functions with absolutely convergent Fourier series

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(k) e^{ikx}, \quad \|f\|_A = \sum_{-\infty}^{\infty} |\hat{f}(k)|.$$

For a non-zero integer s we write

$$f_s(x) = f(sx).$$

Of course,

$$\|f_s\|_A = \|f\|_A.$$

Also, if $A = \operatorname{supp}\hat{f}$, then $\operatorname{supp}\hat{f}_s = sA = \{sa\colon s \in A\}$. Moreover,

$$\operatorname{supp}(f_{s_1} f_{s_2})\hat{} = \operatorname{supp}\hat{f}_{s_1} * \hat{f}_{s_2} \in s_1 A + s_2 A.$$

Consequently, if $s_1, \ldots, s_n$ are such that every element a in $s_1 A + \ldots + s_n A$ has only one presentation in the form

$$a = s_1 a_1 + \ldots + s_n a_n, \quad a_j \in A,$$

then

$$\|f_{s_1} \cdots f_{s_n}\|_A = \|f\|_A^n.$$

From this one can deduce that if for a given function f in $A(T)$ a sequence $s_1, s_2, \ldots$ of integers is lacunary enough, then

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n} \log \|f_{s_1} \ldots f_{s_n}\|_A = \log \|f\|_A.$$

On the other hand, if we take a constant sequence $s_n = 1$, then

$$(2) \qquad \lim_{n\to\infty} \frac{1}{n} \log \|f^n\|_A = \log \|f\|_{L^\infty}.$$

More general easy inequalities of this type are

$$\log|f(0)| \leqslant \liminf_{n\to\infty}\frac{1}{n}\log\|f_{s_1}\dots f_{s_n}\|_A$$
$$\leqslant \limsup_{n\to\infty}\frac{1}{n}\log\|f_{s_1}\dots f_{s_n}\|_A \leqslant \log\|f\|_A.$$

In Section 2 of the present paper we show that, in fact, for every f in $A(T)$ there is a sequence $s_1, s_2, \dots$ of integers such that

$$\liminf_{n\to\infty}\frac{1}{n}\log\|f_{s_1}\dots f_{s_n}\|_A \leqslant \log\|f\|_{L^\infty},$$
$$\limsup_{n\to\infty}\frac{1}{n}\log\|f_{s_1}\dots f_{s_n}\|_A = \log\|f\|_A. \tag{3}$$

The aim of this paper is to show that sequences for which (1) or (3) hold are, in fact, quite rare. This is made precise by the following

THEOREM. *Let S be the set of all integer-valued sequences, $S = Z\times Z\times\dots$ There is a subset S_0 of S such that*

(i) *if μ is a probability measure on Z and $\boldsymbol{\mu} = \mu\times\mu\times\dots$ is the direct product measure on S, then $\boldsymbol{\mu}(S_0) = 1$;*

(ii) *for every f in $A(T)$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\|f_{s_1}\dots f_{s_n}\|_A \leqslant \log\|f\|_{L^\infty}$$

for every sequence $s_1, s_2, \dots$ in S_0.

The proof of the Theorem is based on the following

LEMMA. *Let A be a finite subset of Z with $|A| \geqslant 1$, μ a probability measure on Z and $c > 1$. We write*

$$B_c = \{s\in S\colon s \text{ has a subsequence } s_{n_k} \text{ s.t. } |s_1A + \dots + s_{n_k}A| > c^{n_k}\}$$

and

$$B = \bigcup_{c>1} B_c = \bigcup_{n=1}^{\infty} B_{1+1/n}.$$

Then $\boldsymbol{\mu}(B) = 0$.

COROLLARY. *There is a subset S_0 of S such that for every non-empty finite subset A of Z*

$$\limsup_{n\to\infty}\frac{1}{n}\log|s_1A + \dots + s_nA| = 0$$

for all $s = (s_1, s_2, \dots)$ in S_0, and $\boldsymbol{\mu}(S_0) = 1$ for every probability measure μ on Z.

Proof of the Lemma. For a given number r $(0 < r < 1)$ and a finite set T in Z we define a subset of S as follows:

$$M_n(r, T) = \{s \in S: \text{ at most } nr \text{ among } s_1, \ldots, s_n \text{ are not in } T\}.$$

For s in $M_n(r, T)$ we prove

$$|s_1 A + \ldots + s_n A| \leqslant (n+1)^{|A||T|} |A|^{rn}. \tag{4}$$

In fact, for an element a in the set of the left-hand side of (4) we write

$$a = \sum_{i \leqslant n} s_i a_i = \sum_{\substack{i \leqslant n \\ s_i \in T}} s_i a_i + \sum_{\substack{i \leqslant n \\ s_i \notin T}} s_i a_i. \tag{5}$$

The number of integers which can be presented as the first summand in (5) is at most $(n+1)^{|A||T|}$. On the other hand, the number of summands in the second summand in (4) is at most nr; therefore, for a fixed s at most $|A|^{nr}$ numbers can be presented in this form. This proves (4).

Now we take an arbitrary $c > 1$ and we choose r in such a way that $|A|^r < c^{1/2}$. Then for all T and n we have

$$|s_1 A + \ldots + s_n A| \leqslant (n+1)^{|A||T|} c^{n/2} \tag{6}$$

for every s in $M_n(r, T)$.

Further, r being fixed, we select a subset T of Z so large that

$$\mu(Z \setminus T)^r < \tfrac{1}{2}|A|^{-1}.$$

Then for every n we have

$$\mu(S \setminus M_n(r, T)) < |A|^{-n}. \tag{7}$$

In fact,

$$\begin{aligned} \mu(S \setminus M_n(r, T)) &= \text{the probability of selecting more than } rn \\ &\quad\ \text{elements from } Z \setminus T \text{ in } n \text{ independent trials} \\ &= \sum_{rn < k \leqslant n} \binom{n}{k} \mu(T)^{n-k} \mu(Z \setminus T)^k \\ &\leqslant [2\mu(Z \setminus T)^r]^n. \end{aligned}$$

Let

$$M_m = \bigcap_{n=m}^{\infty} M_n(r, T).$$

If $s \in M_m$ for a fixed m, then for every $n \geqslant m$ inequality (6) holds and, consequently,

$$B_c \subset S \setminus M_m = \bigcup_{n=m}^{\infty} [S \setminus M_n(r, T)].$$

Hence, by (7),

$$\mu(B_c) \leqslant \sum_{n=m}^{\infty} |A|^{-n}$$

for all m, and so the proof of the Lemma is completed.

Proof of the Theorem. Suppose first that $\varphi \in A(T)$ is a trigonometric polynomial, i.e. $\operatorname{supp}\hat{\varphi} = A$ is finite. Then for an arbitrary s in S we have

$$\begin{aligned}
\|\varphi_{s_1} \ldots \varphi_{s_n}\|_A &= \|\hat{\varphi}_{s_1} * \ldots * \hat{\varphi}_{s_n}\|_{l^1} \\
&\leqslant \|\hat{\varphi}_{s_1} * \ldots * \hat{\varphi}_{s_n}\|_{l^2} |s_1 A + \ldots + s_n A|^{1/2} \\
&= \|\varphi_{s_1} \ldots \varphi_{s_n}\|_{L^2} |s_1 A + \ldots + s_n A|^{1/2} \\
&\leqslant \|\varphi_{s_1} \ldots \varphi_{s_n}\|_{L^\infty} |s_1 A + \ldots + s_n A|^{1/2}.
\end{aligned}$$

To prove the Theorem we may assume $\|f\|_{L^\infty} = 1$. Then we take an arbitrary $c > 1$ and let

$$S_c = \left\{ s \in S: \limsup_{n \to \infty} \frac{1}{n} \log \|f_{s_1} \ldots f_{s_n}\|_A < \log c \right\}.$$

It suffices to prove that $\mu(S_c) = 1$. By the Corollary, this is implied by $S_c \supset S_0$, which we now prove.

Let $\varepsilon > 0$ be such that $1 + 2\varepsilon < c^{1/2}$ and we take a trigonometric polynomial ψ such that $f = \psi + g$ with $\|g\|_A < \varepsilon$. Then, of course,

$$\|\psi\|_{L^\infty} < 1 + \varepsilon.$$

Let $s \in S_0$. If $A = \operatorname{supp}\hat{\psi}$, then for n sufficiently large

$$|s_{i_1} A + \ldots + s_{i_k} A| \leqslant |s_1 A + \ldots + s_n A| < c^n$$

for all $1 \leqslant i_1 < \ldots < i_k \leqslant n$. Hence, if

$$\{\{i_1, \ldots, i_k\}: 1 \leqslant i_1 < \ldots < i_k \leqslant n\} = \left\{ i_k^{(r)}: 1 \leqslant r \leqslant \binom{n}{k} \right\},$$

then

$$\begin{aligned}
\|f_{s_1} \ldots f_{s_n}\|_A &= \|(\psi + g)_{s_1} \ldots (\psi + g)_{s_n}\|_A \\
&\leqslant \sum_{k=0}^{n} \sum_{r=1}^{\binom{n}{k}} \Big\| \prod_{j \in i_k^{(r)}} \psi_{s_j} \Big\|_A \Big\| \prod_{\substack{j \leqslant n \\ j \notin i_k^{(r)}}} g_{s_j} \Big\|_A \\
&\leqslant \sum_{k=0}^{n} \sum_{r=1}^{\binom{n}{k}} \|\psi\|_{L^\infty}^k |s_{i_1} A + \ldots + s_{i_k} A|^{1/2} \varepsilon^{n-k} \\
&\leqslant c^{n/2} \sum_{k=0}^{n} \binom{n}{k} (1 + \varepsilon)^k \varepsilon^{n-k} < c^n,
\end{aligned}$$

which completes the proof of the Theorem.

2. Remarks.

Remark 1. *For every function f in $A(T)$ there is a sequence $s_1, s_2, \ldots$ of integers such that* (3) *holds.*

Proof. Of course, we may assume that $\|f\|_{L^\infty} = 1$. First we note that if $g,\ h \in A(T)$, then

(8) $$\lim_{s\to\infty} \|gh_s\|_A = \|g\|_A \|h\|_A$$

(cf. [2], p. 82).

Using (2) and (8) we define sequences $s_1, s_2, \ldots$ and $k_1, k_2, \ldots$ by induction as follows:

$$s_1 = k_1 = 1.$$

Suppose $k_1,\ k_2, \ldots, k_{2m-1}$ and $s_1,\ s_2, \ldots, s_{k_{2m-1}}$ are defined. We let

$$g = f_{s_1} f_{s_2} \cdots f_{s_{k_{2m-1}}},$$

we take n so large that

$$\frac{1}{n}\log(\|g\|_A \|f^n\|_A) \leqslant \frac{1}{m},$$

and we let

$$k_{2m} = k_{2m-1} + n \quad \text{and} \quad s_{k_{2m-1}+1} = \ldots = s_{k_{2m-1}+n} = 1.$$

We then have

(9) $$k_{2m}^{-1} \log \|f_{s_1} \cdots f_{s_{k_{2m}}}\|_A \leqslant 1/m.$$

Now, suppose $k_1, \ldots, k_{2m}$ and $s_1,\ s_2, \ldots, s_{k_{2m}}$ are defined. We write

$$g^{(0)} = f_{s_1} \cdots f_{s_{k_{2m}}}$$

and we put $k_{2m+1} = k_{2m} + n$, where n is so large that

$$\frac{1}{k_{2m}+n} \log\left[(1-1/m)\|g^{(0)}\|_A \|f\|_A^n\right] \geqslant \log\|f\|_A (1-1/m).$$

Now we proceed by induction for $j \leqslant n$ letting, by (8), $s_{k_{2m}+j}$ be such that if

$$g^{(j)} = g^{(j-1)} f_{s_{k_{2m}+j}},$$

then

$$\|g^{(j)}\|_A \geqslant \|g^{(j-1)}\|_A \|f\|_A (1-1/m)^{1/n}.$$

Hence

$$\|g^{(n)}\|_A \geqslant \|g^{(0)}\|_A \|f\|_A^n (1-1/m)$$

and, consequently,

(10) $$k_{2m+1}^{-1}\log\|f_{s_1}\dots f_{s_{k_{2m+1}}}\|_A \geqslant \log\|f\|_A(1-1/m).$$

Thus, by (9) and (10), the proof is completed.

Remark 2. The estimate given by the Theorem is sharp. In fact, it is known (it follows from [2], pp. 81–82) that for every sequence of positive numbers ε_n which tends to zero there is a function f in $A(T)$ such that

$$\frac{1}{n}\log\|f^n\|_A \geqslant \log\|f\|_{L^\infty}+\varepsilon_n.$$

However, we do not know whether for $\varepsilon_n \searrow 0$ there are a trigonometric polynomial φ and a probability measure μ on Z such that

$$\frac{1}{n}\log\|\varphi_{s_1}\dots\varphi_{s_n}\|_A \geqslant \log\|\varphi\|_A+\varepsilon_n$$

for μ-almost all sequences $s_1, s_1, \dots$ (**P 1349**). We note that for trigonometric polynomials with $\|\varphi\|_{L^\infty} = 1$ we have (cf. [1])

$$\|\varphi^n\|_A \leqslant C \quad \text{for } n = 1, 2, \dots$$

The authors are grateful to O. Carruth McGehee for interesting discussions and information related to the contents of this paper.

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Reçu par la Rédaction le 9. 10. 1985