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IDEMPOTENT WORDS IN NILPOTENT GROUPS

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0. A word $w(x_1, \ldots, x_n)$ in a group G is called *idempotent* if the equality $w(x, \ldots, x) = x$ holds for every $x \in G$.

In this note we prove the following

THEOREM. *Every idempotent word in a nilpotent group is a superposition of the words xyz^{-1} and x.*

1. Let us recall the identities

$$[xy, z] = [x, z][x, z, y][y, z] \tag{1}$$

and

$$[x, yz] = [x, z][x, y][x, y, z] \tag{2}$$

which hold for every group G. From (2) we have

$$1 = [x, yy^{-1}] = [x, y^{-1}][x, y][x, y, y^{-1}],$$

whence we get another well-known identity

$$[x, y^{-1}]^{-1} = [x, y][x, y, y^{-1}]. \tag{3}$$

The set of all words which are superpositions of the words xyz^{-1} and x we denote by $\{\overline{xyz^{-1}}\}$.

2. LEMMA 1. *In a nilpotent group we have $xy^{-1}z \in \{\overline{xyz^{-1}}\}$.*

Proof. We have

$$y[z^{-1}, y]^{-1} = zyz^{-1}$$

and

$$y[z^{-1}, y, y^{-1}] = y[z^{-1}, y]^{-1}y[z^{-1}, y]y^{-1} = (y[z^{-1}, y]^{-1})y(y[z^{-1}, y]^{-1})^{-1}.$$

Hence $y[z^{-1}, y]^{-1}$ and $y[z^{-1}, y, y^{-1}]$ belong to $\{\overline{xyz^{-1}}\}$.

Observe that if $yk \in \{\overline{xyz^{-1}}\}$, then $y[k^{-1}, y^{-1}] \in \{\overline{xyz^{-1}}\}$. Indeed, we have $y[k^{-1}, y^{-1}] = yky k^{-1}y^{-1} = yky(yk)^{-1}$.

Let us write

$$u_0 = [z^{-1}, y, y^{-1}] \quad \text{and} \quad u_{n+1} = [u_n^{-1}, y^{-1}] \quad \text{for } n \geqslant 0.$$

Hence we see that all members of the sequence yu_n, $n = 0, 1, \ldots$, are in $\{\overline{xyz^{-1}}\}$. We have also

$$\begin{aligned} xz^{-1}yu_0 &= xz^{-1}y[z^{-1}, y, y^{-1}] = xyz^{-1}[z^{-1}, y][z^{-1}, y, y^{-1}] \\ &= xyz^{-1}y[z^{-1}, y]y^{-1} = (xyz^{-1})y(y[z^{-1}, y]^{-1})^{-1} \in \{\overline{xyz^{-1}}\}, \\ xz^{-1}yu_{n+1} &= xz^{-1}y[u_n^{-1}, y^{-1}] = (xz^{-1}yu_n)y(yu_n)^{-1}. \end{aligned}$$

Using the induction on n, we see that all members of the sequence $xz^{-1}yu_n$, $n = 0, 1, \ldots$, belong to $\{\overline{xyz^{-1}}\}$.

Since the group is nilpotent, say, of class N, we obtain $u_n = 1$ for $n > N$. This yields $xz^{-1}y \in \{\overline{xyz^{-1}}\}$. But $xy^{-1}z$ is a superposition of $xz^{-1}y$ and

$$e_3^1(x, y, z) = x, \quad e_3^2(x, y, z) = y, \quad e_3^3(x, y, z) = z.$$

Thus, the proof of the lemma is complete.

LEMMA 2. *In a nilpotent group we have* $y^{-1}xz \in \{\overline{xyz^{-1}}\}$.

Proof. From the identities

$$z[y^{-1}, x] = z(yxy^{-1})^{-1}x, \quad z[w, z^{-1}] = zw^{-1}z(zw^{-1})^{-1},$$

$$z[w^{-1}, z^{-1}] = zwz(zw)^{-1}, \quad zw^{-1} = z(zw)^{-1}z \quad \text{and} \quad zw'w = zw'(zw^{-1})^{-1}z$$

we infer, by Lemma 1, that $z[y^{-1}, x]$, zw^{-1}, $zw'w$, $z[w, z^{-1}]$ belong to $\{\overline{xyz^{-1}}\}$, provided zw, zw' do.

Hence $zv_n \in \{\overline{xyz^{-1}}\}$, $n = 0, 1, \ldots$, where $v_0 = [y^{-1}, x, z^{-1}]^{-1}$ and $v_{n+1} = [v_n, z^{-1}]$ for $n = 0, 1, \ldots$

Let us put $c_0 = [y^{-1}, x, z]$ and $c_{n+1} = [c_n, z^{-1}]$ for $n = 0, 1, \ldots$ We check that $v_n = c_n c_{n+1}$ for $n = 0, 1, \ldots$ It follows from (3) that $v_0 = c_0 c_1$, while equality (1) gives

$$v_{n+1} = [v_n, z^{-1}] = [c_n c_{n+1}, z^{-1}] = [c_n, z^{-1}][c_{n+1}, z^{-1}] = c_{n+1}c_{n+2}.$$

Since the group is nilpotent of class, say, N, one can express c_0 as a word W on "variables" $v_0, \ldots, v_N$. Thus

$$zc_0^{-1}[y^{-1}, z]^{-1} = zW^{-1}[y^{-1}, z]^{-1} \in \{\overline{xyz^{-1}}\}.$$

Finally, we get

$$y^{-1}xz = xy^{-1}z[y^{-1}, x][y^{-1}, x, z] = xy^{-1}z(zc_0^{-1}[y^{-1}, z]^{-1})^{-1}z,$$

which shows that $y^{-1}xz \in \{\overline{xyz^{-1}}\}$.

3. The proof of the theorem follows now by [2], where it is shown that in an arbitrary group every idempotent word is a superposition of the words $x^{-1}yz$, xyz^{-1} and x.

REFERENCES

[1] M. Hall, *The theory of groups*, New York 1959.
[2] J. Płonka, *On the arity of idempotent reducts of groups*, Colloquium Mathematicum 21 (1970), p. 35-37.

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