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LIPSCHITZ EXTENSIONS AND LIPSCHITZ RETRACTIONS IN METRIC SPACES

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The problem of extensions of R^1-valued Lipschitz functions was investigated, e.g., by McShane [6], Banach [2], Mickle [7], Valetine [10]-[12]. Cipszer and Geher [4] and Aronszajn and Panitchpakdi [1] considered the extensions of Lipschitz maps with the values in hyperconvex metric spaces. In this note we study the extensions of Lipschitz maps between metric spaces.

In Section 1 we give some examples of L^λ-spaces (in the sense of [8]). It is shown that there exist L^1-spaces which are not topologically complete. Section 2 is devoted to a proof of a gluing theorem for L-spaces corresponding to Borsuk's theorem on the union of two ANR($\mathfrak{M}$)-spaces.

1. L^λ-spaces. We recall first the following definition [8]:

1.1. Definition. Let $\lambda \in [1, \infty)$. A metric space Y is called an *L^λ-space* if, whenever X is a metric space and A is a closed subset of X, every Lipschitz map $f\colon A \to Y$ can be extended to a Lipschitz map $\tilde{f}\colon X \to Y$ such that $\|\tilde{f}\| \leqslant \lambda \|f\|$. Here $\|\cdot\|$ denotes the Lipschitz norm, i.e.

$$\|g\| = \inf\{K\colon d_Y(g(x), g(y)) \leqslant K d_X(x, y) \text{ for all } x, y \in X\}$$

for any Lipschitz map $g\colon (X, d_X) \to (Y, d_Y)$.

We say that Y is an *L-space* if it is an L^λ-space for some $\lambda \geqslant 1$.

Given any set D, let $l^\infty(D)$ denote the Banach space of all bounded real functions on D with the supremum norm. It is well known that $l^\infty(D)$ is an L^1-space (see [1] and [4]). In this section we give some examples of L^λ-spaces. We prove first the following

1.2. LEMMA. *A metric space Y is an L^λ-space if and only if for every metric space Z containing Y isometrically as a closed subset there exists a Lipschitz retraction from Z onto Y of the norm less than or equal to λ.*

Proof. Let $f\colon A \to Y$ be a Lipschitz map from a closed subset A of a metric space X into Y. Consider Y as a subset of the space $E = l^\infty(Y)$.

Since $l^\infty(Y)$ is an L^1-space, there exists a Lipschitz map $g\colon X \to E$ such that $g|A = f$ and $\|g\| = \|f\|$.

Equip $F = E \times R^1$ with the max-norm and write

$$h(x) = \big(g(x),\ \varrho(x, A)\big) \in F \quad \text{for every } x \in X.$$

Finally, put $Z = h(X) \cup Y \times \{0\}$ and identify Y with $Y \times \{0\}$. Clearly, Y is closed in Z. Let $r\colon Z \to Y$ be a Lipschitz retraction of the norm less than or equal to λ. Then $\tilde{f} = rh$ is the required Lipschitz extension of f.

Metric spaces X and Y are called *Lipschitz equivalent* if there exists a one-to-one map f from X onto Y such that both f and f^{-1} satisfy the Lipschitz condition. It is clear that

1.3. PROPOSITION. *If X and Y are Lipschitz equivalent metric spaces and X is an L-space, then Y is an L-space.*

Let us prove the following

1.4. PROPOSITION. *Every closed convex set lying in a Minkowski space E^n is an L-space.*

Proof. It is clear that E^n is Lipschitz equivalent to $l^2(n)$ and to $l^\infty(n)$, the latter being an L^1-space. Therefore, we infer the result from the following

1.5. LEMMA. *If X is a closed convex set lying in a Hilbert space H, then the nearest-point retraction satisfies the Lipschitz condition with constant* 1.

Proof. For every point $x \in H$, let $p(x)$ be the nearest point of X. Since X is convex, $p(x)$ is uniquely defined. Given $x_1, x_2 \in H$ with $p(x_1) \neq p(x_2)$, consider points y_1, y_2 of the intersection of the line passing through x_1 and x_2 with the hyperplane passing through $p(x_1)$ (respectively, $p(x_2)$) and perpendicular to the segment $[p(x_1), p(x_2)]$. Since

$$\|z - x_1\| \geqslant \|p(x_1) - x_1\| \quad \text{for } z \in [p(x_1), p(x_2)],$$

we have $\sphericalangle\big(p(x_2), p(x_1), x_1\big) \geqslant 90°$, and hence y_1 (similarly, y_2) lies between x_1 and x_2. Thus

$$\|x_1 - x_2\| \geqslant \|y_1 - y_2\| \geqslant \|p(x_1) - p(x_2)\|$$

as required.

1.6. PROPOSITION. *The open interval $(0, 1)$ is an L^1-space.*

Proof. Let X be a metric space containing $(0, 1)$ isometrically as a closed subset. For every $n = 0, 1, \ldots$ put

$$A_n = \big\{x \in X\colon\ \varrho\big(x,\ (0,\ 1)\big) \geqslant 2^{-n-1}\big\}.$$

We shall construct inductively a sequence of maps $\{f_n\}$ from A_n into $I_n = [2^{-n-1}, 1-2^{-n-1}]$ with the following properties:

(a_n) $f_n = f_{n+1}|A_n$ for every $n = 0, 1, \ldots$,

(b_n) $|z-f_n(x)| \leqslant \varrho(z, x)$ for every point $x \in A_n$ and $z \in (0, 1)$,

(c_n) $|f_n(x)-f_n(y)| \leqslant \varrho(x, y)$ for every $x, y \in A_n$.

Let $f_0(x) = 2^{-1}$ for every $x \in A_0$ and assume that $f_n\colon A_n \to I_n$ satisfying (a_n), (b_n), and (c_n) has been constructed for some $n \geqslant 0$.

Consider the set $\mathscr{B}$ of all pairs (B, f_B) with $A_n \subset B \subset A_{n+1}$ and $f_B : B \to I_{n+1}$ satisfying the following conditions:

(d_n) $f_B|A_n = f_n$,

(e_n) $|z-f_B(x)| \leqslant \varrho(z, x)$ for every $x \in B$ and $z \in (0, 1)$,

(f_n) $|f_B(x)-f_B(y)| \leqslant \varrho(x, y)$ for every $x, y \in B$.

Under the natural partial ordering, by the Kuratowski-Zorn lemma there is a maximal element $(\bar{B}, f_{\bar{B}})$ of $\mathscr{B}$. Let us show that $\bar{B} = A_{n+1}$.

Suppose not and take $x_0 \in A_{n+1} \setminus \bar{B}$. Put

$$m = \max\big\{\sup\{f_{\bar{B}}(x)-\varrho(x, x_0)\colon x \in \bar{B}\},\ \sup\{z-\varrho(z, x_0)\colon z \in (0, 1)\}\big\},$$
$$M = \min\big\{\inf\{f_{\bar{B}}(x)+\varrho(x, x_0)\colon x \in \bar{B}\},\ \inf\{z+\varrho(z, x_0)\colon z \in (0, 1)\}\big\}.$$

It is clear that $m \leqslant M$. Since $x_0 \in A_{n+1}$, we have $m \leqslant 1-2^{-n-2}$ and $M \geqslant 2^{-n-2}$. Therefore, there exists a point $y_0 \in I_{n+1} \cap [m, M]$. Let us define $f\colon \bar{B} \cup \{x_0\} \to I_{n+1}$ by

$$f(x) = \begin{cases} f_{\bar{B}}(x) & \text{if } x \in B, \\ y_0 & \text{if } x = x_0. \end{cases}$$

It is easy to check that $(\bar{B} \cup \{x_0\}, f) \in \mathscr{B}$, contrary to the maximality of $(\bar{B}, f_{\bar{B}})$. Thus $\bar{B} = A_{n+1}$ and $f_{n+1} = f_{\bar{B}}$ satisfies (a_{n+1}), (b_{n+1}), and (c_{n+1}).

We define a retraction $r\colon X \to (0, 1)$ by

$$r(x) = \begin{cases} x & \text{if } x \in (0, 1), \\ \lim\limits_{n\to\infty} f_n(x) & \text{if } x \in X \setminus (0, 1). \end{cases}$$

It is easy to see that $|r(x)-r(y)| \leqslant \varrho(x, y)$ for every $x, y \in X$. Thus the proposition is proved.

Remark. By a theorem of Luukkainen [5] it follows that $(0, 1)$ is an L^λ-space. Proposition 1.6 shows that one can take $\lambda = 1$.

1.7. COROLLARY. *$S = \{x \in l^\infty(D)\colon \|x\| < 1\}$ is an L^1-space.*

1.8. PROPOSITION. *If Y is an L^λ-space, then every metric space Z containing Y isometrically as a dense set is an L^λ-space.*

Proof. Let X be a metric space containing Z isometrically as a dense set. Let $X' = X \setminus (Z \setminus Y)$. Then Y is a closed subset of X', and so there exists a Lipschitz retraction $r'\colon X' \to Y$ of the norm less than or equal to λ. Extending r' by identity over $Z \setminus Y$ we get a retraction $r: X \to Z$ of the Lipschitz norm less than or equal to λ. Corollary 1.7 and Proposition 1.8 imply the existence of incomplete L^1-spaces.

1.9. Example. In the plane R^2 consider the set

$$Y = \{(x, y) \in R^2: \max\{|x|, |y|\} < 1\} \cup$$
$$\cup \{(x, y) \in R^2: \max\{|x|, |y|\} = 1 \text{ and } x, y \text{ are rational}\}.$$

Then, by Corollary 1.7 and Proposition 1.8, the set Y is an L^1-space; however, Y is not of type G_δ in R^2 and, therefore, Y is not complete-metrizable.

2. The union of two L-sets.

2.1. THEOREM. *Let X_0, X_1, X_2 be subsets of a metric space (X, ϱ) such that $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Suppose that X_0, X_1, and X_2 are L-spaces. Then X is an L-space if and only if there exists a constant $K > 0$ such that the metric d on X defined by*

(2.1)
$$d(x, y) = \begin{cases} \varrho(x, y) & \text{if } x, y \in X_i \text{ for } i = 1, 2, \\ \inf\{\varrho(x, z) + \varrho(y, z): z \in X_0\} & \text{otherwise} \end{cases}$$

satisfies the condition

$$d(x, y) \leqslant K\varrho(x, y) \quad \text{for every } x, y \in X. \tag{2.2}$$

Proof. Assume that the metric d defined by (2.1) satisfies (2.2). It can easily be seen that $\bar{X}_0 = \bar{X}_1 \cap \bar{X}_2$. Therefore, by Proposition 1.8, we may assume that X_0, X_1, and X_2 are closed in X. Since the identity map $\mathrm{id}_X: (X, \varrho) \to (X, d)$ is a Lipschitz equivalence by Proposition 1.3, it suffices to prove that, for every metric space (Z, d) containing (X, d) isometrically as a closed subset, Z Lipschitz retracts onto X.

By a theorem of Kuratowski and Wojdysławski [3] we assume without loss of generality that Z is a convex set lying in a normed space. Put

$$Z_1 = \{z \in Z: d(z, X_1) \geqslant d(z, X_2)\}, \quad Z_2 = \{z \in Z: d(z, X_1) \leqslant d(z, X_2)\},$$
$$Z_0 = Z_1 \cap Z_2.$$

Then $Z = Z_1 \cup Z_2$ and X_0 is closed in Z_0. Since X_0 is an L-space, there exists a retraction $r_0: Z_0 \to X_0$ such that

$$d(r_0(x), r_0(y)) \leqslant \lambda d(x, y) \quad \text{for } x, y \in Z_0,$$

where $\lambda < \infty$. For $i = 1, 2$ define a map f_i from $Z_0 \cup X_i$ onto X_i by

$$f_i(z) = \begin{cases} r_0(z) & \text{if } z \in Z_0, \\ z & \text{if } z \in X_i. \end{cases}$$

Let us check that f_i is a Lipschitz map. Let $z \in Z_0$ and $x \in X_1$. By the definition of Z_0 there exists a point $y \in X_2$ such that $d(z, y) \leqslant 2d(z, x)$. Then, necessarily, $d(x, y) \leqslant 3d(x, z)$. By the definition of d there is a point $p \in X_0$ such that

$$d(x, p) + d(p, y) \leqslant 2d(x, y).$$

Then we have

$$\begin{aligned} d(f_1(x), f_1(z)) &= d(x, r_0(z)) \leqslant d(x, p) + d(p, r_0(z)) \\ &\leqslant d(x, p) + \lambda d(p, z) \leqslant d(x, p) + \lambda d(p, y) + \lambda d(y, z) \\ &\leqslant \lambda(2d(x, y) + 2d(y, z)) \leqslant 8\lambda d(x, z). \end{aligned}$$

Thus f_1 and, similarly, f_2 are Lipschitz maps.

Since $X_i \in L$ and $X_i \cup Z_0$ is closed in Z_i, there are Lipschitz maps $\tilde{f}_i\colon Z_i \to X_i$ such that $\tilde{f}_i | X_i \cup Z_0 = f_i$ and

$$d(\tilde{f}_i(x), \tilde{f}_i(y)) \leqslant \lambda_i d(x, y) \quad \text{for } x, y \in Z_0 \text{ and } i = 1, 2.$$

We now define a retraction $r\colon Z \to X$ by setting $r(z) = \tilde{f}_i(z)$ for $z \in X_i$.

Given $x \in Z_1$ and $y \in Z_2$, by the definition of Z_i's there is a $t \in [0, 1]$ such that $z = tx + (1-t)y \in Z_0$. Then

$$\begin{aligned} d(r(x), r(y)) &\leqslant d(r(x), r(z)) + d(r(z), r(y)) = d(f_1(x), f_1(z)) + d(f_2(z), f_2(y)) \\ &\leqslant \lambda_1 d(x, z) + \lambda_2 d(z, y) \leqslant \max\{\lambda_1, \lambda_2\} d(x, y). \end{aligned}$$

Therefore, r is the desired Lipschitz retraction of Z onto X.

Conversely, assume that $(X, \varrho) \in L$. We will show that there is a constant $K > 0$ satisfying condition (2.2).

Consider again (X, ϱ) as a closed subset of a convex set lying in a normed space. Let $r\colon Z \to X$ be a Lipschitz retraction such that $\varrho(r(x), r(y)) \leqslant K\varrho(x, y)$ for all $x, y \in Z$. Let $x \in X_1$ and $y \in X_2$. It is easy to see that there is a $z \in [x, y]$ such that $r(z) \in X_0$. Then we have

$$d(x, y) \leqslant \varrho(x, r(z)) + \varrho(r(z), y) \leqslant K\varrho(x, z) + K\varrho(z, y) = K\varrho(x, y).$$

Thus the theorem is proved.

2.2. COROLLARY. *In the notation of Theorem 2.1, if X and X_0 are L-spaces, then so are X_1 and X_2.*

Proof. Let us note that if $(X, \varrho) \in L$, then it follows from the proof of Theorem 2.1 that the metric d defined by (2.1) satisfies (2.2), and hence is Lipschitz equivalent to ϱ.

Now let Z be a metric space containing (X_1, ϱ) isometrically as a closed subset. Since X is an L-space, there exists a Lipschitz map $f\colon Z \to X$ such that $f | X_1 = \mathrm{id}_{X_1}$. Let $r'\colon X_2 \to X_0$ be a Lipschitz retraction and define $r\colon Z \to X_1$ by

$$r(x) = \begin{cases} f(x) & \text{if } f(x) \in X_1, \\ r'f(x) & \text{if } f(x) \in Z \setminus X_1. \end{cases}$$

Since ϱ and d are Lipschitz equivalent, an easy computation shows that r is a Lipschitz map. Thus X_1 (and similarly X_2) is an L-space.

Remark. The proofs of Theorem 2.1 and Corollary 2.2 are similar to those in [9].

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