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TO CHARLES LOEWNER
WITH ADMIRATION

ON CONFORMALITY POINTS OF CERTAIN MAPPINGS PRESERVING LOCALLY THE AREA

BY

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1. A mapping of the closed unit disc K onto itself given by the formulae

$$u = u(x, y), \quad v = v(x, y) \tag{1}$$

will be called a *mapping preserving locally the area on the boundary*, if functions (1) belong to the class C^1 in an open set which includes K and if

$$u_x(x, y)v_y(x, y) - u_y(x, y)v_x(x, y) = 1 \tag{2}$$

for all points belonging to the boundary K^* of K.

A point (x_0, y_0) is said to be a *conformality point* of a mapping (1) belonging to C^1, if the derivatives u_x, u_y, v_x, v_y do not vanish simultaneously in (x_0, y_0) and if they satisfy there the Cauchy-Riemann equations

$$u_x(x, y) = v_y(x, y), \tag{3_1}$$

$$-u_y(x, y) = v_x(x, y). \tag{3_2}$$

A mapping (1), not necessarily belonging to the class C^1, will be called a *mapping preserving the boundary*, if for any point (x, y) of the circle K^* we have

$$u(x, y) = x, \tag{4_1}$$

$$v(x, y) = y. \tag{4_2}$$

The present paper concerns the problem of existence of conformality points for mappings transforming the disc K onto itself, preserving the boundary and locally the area on the boundary. We call them *J-mappings*. The fundamental result of this paper is formulated in Theorem 3.

2. Put

$$P = \{(x, y) \in K^* : v_x(x, y) = u_y(x, y) = 0\}.$$

We shall prove

LEMMA 1. *If* (1) *is a* J*-mapping, then the only meet points of the curve* (3_1) *with* K^* *are the points*

$$(5_1)\qquad (1,0),\quad (-1,0),\quad (0,1),\quad (0,-1)$$

and all points of the set P, *while the only meet points of the curve* (3_2) *with* K^* *not belonging to* P *are*

$$(5_2)\qquad (2^{-1/2},2^{-1/2}),\quad (-2^{-1/2},-2^{-1/2}),\quad (2^{-1/2},-2^{-1/2}),\quad (-2^{-1/2},2^{-1/2}).$$

Proof. If $x\neq 0$, then from (4) we get

$$u_x x_y+u_y=x_y,\qquad v_x x_y+v_y=1,\qquad x x_y+y=0,$$

for $(x,y)\in K^*$, whence

$$x_y=-y/x,\qquad u_y=-(y/x)(1-u_x),\qquad v_y=1+(y/x)v_x,$$

which, in view of (2), yields

$$(6)\qquad u_x=1-(y/x)v_x,\qquad u_y=-(y/x)^2v_x,\qquad v_y=1+(y/x)v_x$$

and thus

$$(7)\qquad v_y-u_x=2(1/x)yv_x,\qquad v_x+u_y=(1/x)^2(x+y)(x-y)v_x$$

for all $(x,y)\in K^*$.

Similarly, if $y\neq 0$ and $(x,y)\in K^*$, then from (4) we have

$$u_x+u_y y_x=1,\qquad v_x+v_y y_x=y_x,\qquad x+yy_x=0,$$

whence, in view of (2), we obtain

$$(8)\qquad u_x=1+(x/y)u_y,\qquad v_x=-(x/y)^2u_y,\qquad v_y=1-(x/y)u_y$$

and, consequently,

$$(9)\qquad v_y-u_x=-2(1/y)xu_y,\qquad v_x+u_y=-(1/y)^2(x+y)(x-y)u_y.$$

Relations (7) and (9) prove Lemma 1.

For any J-mapping and every $(x,y)\in K^*$ we get from (6) and (8):

1° if $x\neq 0$ and $y\neq 0$, then $v_x=0$ is equivalent to $u_y=0$,

2° if $x\neq 0$ and $y=0$, then $u_y=0$,

3° if $x=0$ and $y\neq 0$, then $v_x=0$.

From Lemma 1 we get also immediately

THEOREM 1. *If* (1) *is a* J*-mapping, then each point of the set* P *is a conformality point of* (1).

In this way the problem has been solved easily when the set P is not empty. The opposite case, which is essential, will be considered in the sequel.

3. In this part we shall consider branches of curves (3_1) and (3_2) passing through the points (5_1) and (5_2), respectively (comp. Fig. 1).

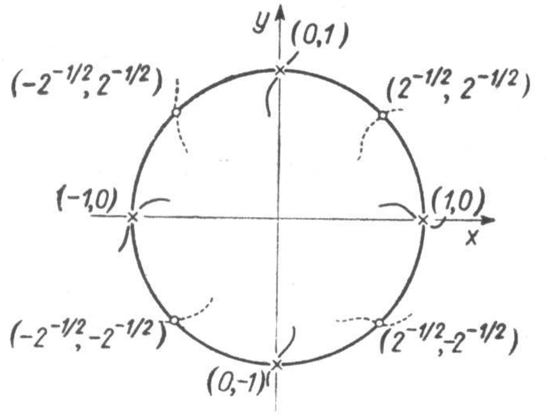

Fig. 1

LEMMA 2. *If (1) is a J-mapping of the class C^2 and if the set P is empty, then each of the points (5_1) is a regular* [1] *point of the curve (3_1) and the branch of curve (3_1), passing through this point, always enters* [2] *the interior of K; similarly, each of the points (5_2) is a regular point of the curve (3_2) and the branch of curve (3_2), passing through this point, always enters the interior of K.*

Proof. To prove the Lemma it is sufficient to verify that in any of the points (5_1) and (5_2) the derivative of the corresponding function, i. e. of $v_y - u_x$ or of $v_x + u_y$, in the direction of one of the semitangents to the circle K^* does not vanish. This derivative may be expressed as

$$(v_y - u_x)_y \text{ for points } (1,0), (-1,0),$$

$$(v_y - u_x)_x \text{ for points } (0,1), (0,-1),$$

$$2^{-1/2}(v_x + u_y)_x - 2^{-1/2}(v_x + u_y)_y \text{ for points } (2^{-1/2}, 2^{-1/2}), (-2^{-1/2}, -2^{-1/2}),$$

$$2^{-1/2}(v_x + u_y)_x + 2^{-1/2}(v_x + u_y)_y \text{ for points } (2^{-1/2}, -2^{-1/2}), (-2^{-1/2}, 2^{-1/2}).$$

Consider at first the curve (3_1) at the points (5_1).

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(1) Under a *regular point* of the curve $F(x, y) = 0$, where $F \in C^1$, we understand here a point at which at least one of the derivatives F_x and F_y does not vanish.

(2) "enters" means here that the tangent to a branch in the corresponding point of K^* is not simultaneously tangent to K^*.

If $x \neq 0$, then from (6) we get

$$\begin{aligned} v_{xy}x_y + v_{yy} &= (1/x)v_x - y(1/x)^2 x_y v_x + y(1/x)v_{xx}x_y + y(1/x)v_{xy}, \\ u_{xx}x_y + u_{xy} &= -(1/x)v_x + y(1/x)^2 x_y v_x - y(1/x)v_{xx}x_y - y(1/x)v_{xy}, \\ u_{xy}x_y + u_{yy} &= -2y(1/x)^2 v_x + 2y^2(1/x)^3 x_y v_x - y^2(1/x)^2 v_{xx}x_y - \\ &\qquad - y^2(1/x)^2 v_{xy}. \end{aligned} \tag{10}$$

Hence putting $y = 0$ and taking in account that (in view of $x_y = -y/x$) $x_y = 0$, we have

$$v_{yy} = (1/x)v_x, \qquad u_{xy} = -(1/x)v_x, \qquad u_{yy} = 0$$

and, in consequence,

$$(v_y - u_x)_y = (2/x)v_x \neq 0. \tag{11}$$

Analogously, for $x = 0$, $y \neq 0$, in virtue of $y_x = -x/y$ and (8) we easily obtain

$$u_{xx} = (1/y)u_y, \qquad v_{xx} = 0, \qquad v_{xy} = -(1/y)u_y$$

and, in consequence,

$$(v_y - u_x)_x = -(2/y)u_y \neq 0. \tag{12}$$

Consider in turn the curve (3_2) at points (5_2).

Suppose at first that $y = x$. Then, in view of $x_y = -y/x$, the system of equations (10) takes the form

$$\begin{aligned} -v_{xy} + v_{yy} &= (2/x)v_x - v_{xx} + v_{xy}, \\ -u_{xx} + u_{xy} &= -(2/x)v_x + v_{xx} - v_{xy}, \\ -u_{xy} + u_{yy} &= -(4/x)v_x + v_{xx} - v_{xy}, \end{aligned}$$

whence

$$2^{-1/2}(v_x + u_y)_x - 2^{-1/2}(v_x + u_y)_y = 2^{1/2}(2/x)v_x \neq 0. \tag{13}$$

Let now $y = -x$. Then, in view of $y_x = -x/y$, we have similarly

$$\begin{aligned} v_{xy} + v_{yy} &= (2/x)v_x - v_{xx} - v_{xy}, \\ u_{xx} + u_{xy} &= -(2/x)v_x + v_{xx} + v_{xy}, \\ u_{xy} + u_{yy} &= (4/x)v_x - v_{xx} - v_{xy}, \end{aligned}$$

whence

$$2^{-1/2}(v_x + u_y)_x + 2^{-1/2}(v_x + u_y)_y = 2^{1/2}(2/x)v_x \neq 0. \tag{14}$$

Thus Lemma 2 is proved.

4. According to Lemma 2 there are four different points on K^* through which regular branches of the curve (3_1) pass and there are four other different points on K^* which are initial points of regular branches of the curve (3_2). There arises the problem of continuability and course of these branches inside of the disc K. Here we shall confine ourselves to the investigation of a certain subclass of the J-mappings.

We distinguish three following subclasses of mappings of the disc K onto itself:

(i) mappings of the class C^2 in K such that the system of the equations

$$u_x(x, y) = v_y(x, y), \tag{15_1}$$

$$u_{xx}(x, y) = v_{xy}(x, y), \tag{15_2}$$

$$u_{xy}(x, y) = v_{yy}(x, y) \tag{15_3}$$

has no solutions in the interior of K, while the systems of (15_1) and (15_2) and of (15_1) and (15_3) have a finite number of solutions in the interior of K,

(ii) mappings of the class C^2 in K such that the system of three equations

$$-u_y(x, y) = v_x(x, y), \tag{16_1}$$

$$-u_{xy}(x, y) = v_{xx}(x, y), \tag{)16_2}$$

$$-u_{yy}(x, y) = v_{xy}(x, y) \tag{16_3}$$

has no solutions in the interior of K, while the systems of (16_1) and (16_2) and of (16_1) and (16_3) have in the interior of K a finite number of solutions,

(iii) mappings that fulfil simultaneously conditions (i) and (ii).

In the sequel mappings (iii) will be called *regular*.

LEMMA 3. *There exist regular J-mappings such that the set P is empty.*

Proof. Consider the uniparametric family of mappings of K onto itself

$$\begin{aligned} u^*(t, x, y) &= x\cos(1-x^2-y^2)-y\sin(1-x^2-y^2)+tx(1-x^2-y^2)^2, \\ v^*(t, x, y) &= x\sin(1-x^2-y^2)+y\cos(1-x^2-y^2)+tx(1-x^2-y^2)^2 \end{aligned} \tag{17}$$

where $t > 0$. It can be easily verified that for sufficiently small t all mappings of this family are J-mappings and that for each of them the set P is empty. There remains to prove that condition (iii) is also fulfilled for any $t > 0$.

At first, we prove that for any $t > 0$, sufficiently small, the systems of equations

$$(18_1) \qquad v_y^*(t, x, y) - u_x^*(t, x, y) = 0,$$

$$(18_2) \qquad v_{xy}^*(t, x, y) - u_{xx}^*(t, x, y) = 0,$$

$$(18_3) \qquad v_{yy}^*(t, x, y) - u_{xy}^*(t, x, y) = 0$$

and

$$(19_1) \qquad v_x^*(t, x, y) + u_y^*(t, x, y) = 0,$$

$$(19_2) \qquad v_{xx}^*(t, x, y) + u_{xy}^*(t, x, y) = 0,$$

$$(19_3) \qquad v_{xy}^*(t, x, y) + u_{yy}^*(t, x, y) = 0$$

have no solutions in the interior of K.

Suppose the contrary. Then there exist numbers $t_n > 0$, $t_n \to 0$ such that one of the systems (18) and (19) has a solution $x = x_n$, $y = y_n$ for $t = t_n$ $(n = 1, 2, \ldots)$. Suppose that it holds for (18); for (19) our reasoning would be analogous. It is possible to assume, choosing eventually subsequences, that the points (x_n, y_n) tend to a certain limit point (x_0, y_0). Then the system (18) is satisfied for $t = 0$, $x = x_0$, $y = y_0$. This, however, implies $x_0 = 0$, $y_0 = 0$, as it is easily seen. Thus by the mean value theorem we have

$$(v_{yt}^* - u_{xt}^*)t_n + (v_{xy}^* - u_{xx}^*)x_n + (v_{yy}^* - u_{xy}^*)y_n = 0,$$

$$(v_{xyt}^* - u_{xxt}^*)t_n + (v_{xxy}^* - u_{xxx}^*)x_n + (v_{xyy}^* - u_{xxy}^*)y_n = 0,$$

$$(v_{yyt}^* - u_{xyt}^*)t_n + (v_{xyy}^* - u_{xxy}^*)x_n + (v_{yyy}^* - u_{xyy}^*)y_n = 0,$$

where the values of partial derivatives are taken at certain points depending on n and situated on the segment connecting $(0, 0, 0)$ and (t_n, x_n, y_n). Hence

$$\begin{vmatrix} v_{yt}^* - u_{xt}^* & v_{xyt}^* - u_{xxt}^* & v_{yyt}^* - u_{xyt}^* \\ v_{xy}^* - u_{xx}^* & v_{xxy}^* - u_{xxx}^* & v_{xyy}^* - u_{xxy}^* \\ v_{yy}^* - u_{xy}^* & v_{xyy}^* - u_{xxy}^* & v_{yyy}^* - u_{xyy}^* \end{vmatrix} = 0.$$

By letting $n \to \infty$ and in view of (18_2) and (18_3) we get

$$(20) \qquad \begin{vmatrix} v_{xxy}^* - u_{xxx}^* & v_{xyy}^* - u_{xxy}^* \\ v_{xyy}^* - u_{xxy}^* & v_{yyy}^* - u_{xyy}^* \end{vmatrix} = 0$$

or

$$(21) \qquad v_{yt}^* - u_{xt}^* = 0,$$

where the values of partial derivatives are taken at the point $(0,0,0)$. Now, from (17) we get at the point $(0,x,y)$

$$\begin{aligned}\tfrac{1}{2}(v_y^*-u_x^*) &= (y^2-x^2)\sin(1-x^2-y^2)-2xy\cos(1-x^2-y^2)\\ &= (y^2-x^2)\sin 1-2xy\cos 1+O(x^4+2x^2y^2+y^4),\end{aligned}$$

$$\begin{aligned}\tfrac{1}{4}(v_y^*-u_x^*)_x &= -x\sin 1-y\cos 1+O(x^2+y^2),\\ \tfrac{1}{4}(v_y^*-u_x^*)_y &= y\sin 1-x\cos 1+O(x^2+y^2).\end{aligned}$$

Therefore at the point $(0,0,0)$ one has

$$(22)\quad (v_y^*-u_x^*)_{xx} = -4\sin 1,\quad (v_y^*-u_x^*)_{xy} = -4\cos 1,\quad (v_y^*-u_x^*)_{yy} = 4\sin 1$$

and, in consequence, the determinant at the left-hand side of (20) is equal to -16. Hence (21) must hold. But $u_t^* = v_t^* = x(1-x^2-y^2)^2$ and so we have $u_{xt}^* = 1$ and $v_{yt}^* = 0$ at the point $(0,0,0)$, which contradicts formula (21).

Let us now prove that for sufficiently small $t>0$ the systems of equations (a) (18_1), (18_2); (b) (18_1), (18_3); (c) (19_1), (19_2); (d) (19_1), (19_3) have at most a finite number of solutions in the interior of K. We prove it only for the system (a); in the remaining cases the reasoning runs analogously.

Suppose that there are numbers $t_n'>0$, $t_n'\to 0$ such that the system (a) has infinitely many different solutions (x_{mn}', y_{mn}'), $m=1,2,\ldots$, inside K and that for every n the points (x_{mn}', y_{mn}') tend to a certain limiting point (x_n', y_n'), and the points (x_n', y_n') to a certain point (x_0', y_0') (we can assume the validity of the latter conditions by eventually choosing subsequences). We shall show at first that $(x_0', y_0')\neq(0,0)$. By the mean value theorem we have

$$\begin{aligned}(v_{xy}^*-u_{xx}^*)(x_{mn}'-x_n')+(v_{yy}^*-u_{xy}^*)(y_{mn}'-y_n') &= 0,\\ (v_{xxy}^*-u_{xxx}^*)(x_{mn}'-x_n')+(v_{xyy}^*-u_{xxy}^*)(y_{mn}'-y_n') &= 0,\end{aligned}$$

where the values of partial derivatives are taken at certain points situated on the segment connecting (t_n', x_n', y_n') and (t_n', x_{mn}', y_{mn}'). Hence

$$\begin{vmatrix} v_{xy}^*-u_{xx}^* & v_{xxy}^*-u_{xxx}^* \\ v_{yy}^*-u_{xy}^* & v_{xyy}^*-u_{xxy}^* \end{vmatrix} = 0.$$

For $m\to\infty$ in view of (18_2) we get $v_{xxy}^* = u_{xxx}^*$ or $v_{yy}^* = u_{xy}^*$, where the values of partial derivatives are taken at the point (t_n', x_n', y_n'). If we had $x_0'=0$ and $y_0'=0$, then $v_{xxy}^*=u_{xxx}^*$ would yield a contradiction with (22) for $n\to\infty$. Similarly, if we had $v_{yy}^*=u_{xy}^*$, then the point

(t_n', x_n', y_n') would satisfy the system of equations (18), which is impossible as it was already proved. Thus $x_0' \neq 0$ or $y_0' \neq 0$. But it can be easily verified that in this case the system of values $t = 0$, $x = x_0'$, $y = y_0'$ does not satisfy the equation (18_3) i. e. at this point $(v_y^* - u_x^*)_y \neq 0$. Hence it follows that in a certain neighbourhood of the point $(0, x_0')$ there exists a function $y = f(t, x)$ analytic in both variables, t and x, and such that the equations

$$v_y^*(t, x, y) - u_x^*(t, x, y) = 0, \qquad y = f(t, x) \tag{23}$$

are equivalent in a certain neighbourhood of the point $(0, x_0', y_0')$. Now, according to the definition of sequences $\{t_n'\}$, $\{(x_{mn}', y_{mn}')\}$, one has

$$v_y^*(t_n', x_{mn}', y_{mn}') - u_x^*(t_n', x_{mn}', y_{mn}') = 0, \tag{24_1}$$

$$v_{xy}^*(t_n', x_{mn}', y_{mn}') - u_{xx}^*(t_n', x_{mn}', y_{mn}') = 0 \tag{24_2}$$

for all indices m, n. Thus, taking into account the properties of these sequences, we obtain from (24_1)

$$y_{mn}' = f(t_n', x_{mn}') \tag{25}$$

for sufficiently large n and m. Hence and from (24_2) it follows

$$v_{xy}^*\big(t_n', x_{mn}', f(t_n', x_{mn}')\big) - u_{xx}^*\big(t_n', x_{mn}', f(t_n', x_{mn}')\big) = 0. \tag{26}$$

Then, putting

$$F_{t_n'}(x) = v_{xy}^*\big(t_n', x, f(t_n', x)\big) - u_{xx}^*\big(t_n', x, f(t_n', x)\big) \tag{27}$$

in a suitable neighbourhood of the point x_0', we see, in view of (26), that for n sufficiently large, the analytic function (27) of one variable x vanishes in a sequence of points x_{mn}' $(m = 1, 2, \ldots)$. Let us prove that for sufficiently large n there are among these points infinitely many different ones. In fact, in the opposite case the sequence $\{x_{mn}'\}$ would contain, for arbitrary large n, infinitely many identic terms $x_{m_j n}'$ $(j = 1, 2, \ldots)$. Consequently, in view of (25), for sufficiently large j the terms

$$y_{m_j n}' = f(t_n', x_{m_j, n}') \qquad (j = 1, 2, \ldots) \tag{28}$$

would be also identic against the assumption that the sequences $\{(x_{mn}', y_{mn}')\}$ $(m = 1, 2, \ldots)$ consist of different terms. Thus, for large n, the (analytic) functions (27) would vanish at infinitely many points, whence they would be identically zero. Therefore, for $n \to \infty$ we had

$$v_{xy}^*\big(0, x, f(0, x)\big) - u_{xx}^*\big(0, x, f(0, x)\big) = 0 \tag{29}$$

in a suitable neighbourhood V of x_0. Moreover, from (23), we had in V

$$v_y^*\big(0, x, f(0, x)\big) - u_x^*\big(0, x, f(0, x)\big) = 0. \tag{30}$$

We easily deduce from (17) that the equations (30) and (29) yield the following identities in V:

$$(f^2-x^2)\sin(1-x^2-f^2)-2xf\cos(1-x^2-f^2) = 0,$$
$$-(x+2x^2f)\sin(1-x^2-f^2)-(f-x^3+xf^2)\cos(1-x^2-f^2) = 0,$$

where $f = f(0, x)$. From each of the above identities it follows that f is a transcendent function. On the other hand, eliminating from them $\sin(1-x^2-f^2)$ and $\cos(1-x^2-f^2)$, we conclude that f is an algebraic function which yields the desired contradiction. Thus the system of equations (18_1), (18_2) has at most a finite number of solutions inside K, and Lemma 3 is proved.

5. In this part we shall solve the problem of continuability of the branches discussed in Lemma 2 inside K for J-mappings which fulfil conditions (i) or (ii).

LEMMA 4. 1° *If* (1) *is a J-mapping which fulfils conditions* (i) *and for which the set P is empty, then equation* (3_1) *is satisfied along two disjoint regular arcs lying inside K except their end points, issuing from* $(1, 0)$ *and* $(-1, 0)$ *respectively and ending either in* $(0, 1)$ *and* $(0, -1)$ *or in* $(0, -1)$ *and* $(0, 1)$.

2° *If* (1) *is a J-mapping which fulfils conditions* (ii) *and for which the set P is empty, then equation* (3_2) *is satisfied along two disjoint regular arcs lying inside K except their end points, issuing from* $(2^{-1/2}, 2^{-1/2})$ *and* $(-2^{-1/2}, -2^{-1/2})$ *respectively and ending either in* $(-2^{-1/2}, 2^{-1/2})$ *and* $(2^{-1/2}, -2^{-1/2})$ *or in* $(2^{-1/2}, -2^{-1/2})$ *and* $(-2^{-1/2}, 2^{-1/2})$.

Proof. We shall prove only the first part, the proof of the second being quite analogous. We shall apply here a method similar to that used by Charzyński [3].

Consider the following system of differential equations:

$$(31)\qquad \begin{aligned} dx/dt &= \varepsilon[v_{yy}(x, y)-u_{xy}(x, y)], \\ dy/dt &= \varepsilon[u_{xx}(x, y)-v_{xy}(x, y)] \end{aligned} \qquad (t \geqslant 0,\ x^2+y^2 \leqslant 1)$$

with the initial conditions

$$(32_1)\qquad x(0) = 1, \quad y(0) = 0$$

or

$$(32_2)\qquad x(0) = -1, \quad y(0) = 0,$$

where the equation $x^2+y^2 = 1$ is admissible only for $t = 0$ and $\varepsilon = 1$ or -1 has to be so chosen that the speed vector $(x'(0), y'(0))$ at the ini-

[3] Z. Charzyński, *Uniformisation des fonctions. Remarque sur les courbes planes*, Bulletin de la Société des Sciences et des Lettres de Łódź IX (8) (1958), p. 1-12, especially p. 3-7.

tial point be directed towards the interior of K. From Lemma 2 it follows that this is always possible (comp. (11)).

We confine ourselves for the moment to the system $(31\text{-}32_1)$. Let us notice that, in view of Lemma 1, this system has a first integral of the form

$$v_y(x,y)-u_x(x,y) = v_y\big(x(0),y(0)\big)-u_x\big(x(0),y(0)\big) = 0. \tag{33}$$

Next we observe that $(31\text{-}32_1)$ has at most one integral solution. In fact, in view of (i) and since (33) is the first integral, the right-hand sides of (31) do not vanish simultaneously along the integral curves of $(31\text{-}32_1)$. Hence we infer immediately that every solution of $(31\text{-}32_1)$ satisfies the equations

$$y = f(x), \tag{34_1}$$

$$dx/dt = \varepsilon\big[v_{yy}\big(x,f(x)\big)-u_{xy}\big(x,f(x)\big)\big] \tag{34_2}$$

or the equations

$$x = g(y), \tag{35_1}$$

$$dy/dt = \varepsilon\big[u_{xx}\big(g(y),y\big)-v_{xy}\big(g(y),y\big)\big] \tag{35_2}$$

in some neighbourhood of any of its points (t_0, x_0, y_0), where the right-hand sides of (34_1) resp. (35_1) denote implicit functions defined by the first integral (33) in a neighbourhood of (x_0, y_0). From the local uniqueness of solutions of the system (34) resp. (35) passing through (t_0, x_0, y_0) we easily deduce the uniqueness of solutions of $(31\text{-}32_1)$ in the local sense and, consequently, also in the integral sense.

Now we see, owing to a well-known existence theorem, that the system $(31\text{-}32_1)$ has exactly one integral solution $x = x_1(t)$, $y = y_1(t)$, $0 \leqslant t < t_1$. Moreover, to distinct values of t' and t'' there correspond distinct values of each of the functions $x_1(t)$ and $y_1(t)$ (see Charzyński, loc. cit. ([3]), p. 5).

We now show that $t_1 < \infty$. In fact, since the systems (15_1), (15_2) and (15_1), (15_3) have only a finite number of solutions, we conclude by (31) that $x_1'(t) \neq 0$ and $y_1'(t) \neq 0$ except for a finite set. Therefore, if our solution were defined for all $t \geqslant 0$, there would exist limits

$$\lim_{t\to\infty} x_1(t) = a, \qquad \lim_{t\to\infty} y_1(t) = b,$$

where $a, b \in K$, and, in view of (33) and (31), we had

$$u_x(a,b) = v_y(a,b) \tag{36}$$

and (see Charzyński, loc. cit. ([3]), p. 6)

$$\lim_{t\to\infty} x_1'(t) = v_{yy}(a, b) - u_{xy}(a, b) = 0,$$

$$\lim_{t\to\infty} y_1'(t) = u_{xx}(a, b) - v_{xy}(a, b) = 0,$$

against condition (i).

Since $t_1 < \infty$ and since the derivatives $x_1'(t), y_1'(t)$ are bounded, there exist limits

$$\lim_{t\to t_1} x_1(t) = x_1(t_1), \qquad \lim_{t\to t_1} y_1(t) = y_1(t_1)$$

and the point $\big(x_1(t_1), y_1(t_1)\big)$ must be situated on the circle K^*. The supposition that

$$x_1(t_1) = x_1(0) = 1, \qquad y_1(t_1) = y_1(0) = 0 \tag{37}$$

leads to a contradiction which can be shown, in account of Lemma 2, by a reasoning analogous to that used by Charzyński, loc. cit. ([3]), p. 7. Hence, as the functions (1) are of the class C^2, we infer from (31) that $x_1(t)$ and $x_2(t)$ are of the class C^1; so they determine a regular arc in the closed interval

$$x = x_1(t), \qquad y = y_1(t) \qquad (0 \leqslant t \leqslant t_1). \tag{38}$$

Notice that according to (33) and (36) equation (3_1) is satisfied along arc (38). So there remains only to investigate the end of this arc.

In virtue of Lemma 1 and (3_1) we have only the following possibilities (as (37) has been eliminated):

$$x_1(t_1) = 0, \qquad y_1(t_1) = 1,$$

$$x_1(t_1) = 0, \qquad y_1(t_1) = -1,$$

$$x_1(t_1) = -1, \qquad y_1(t_1) = 0.$$

We show that the third of them cannot occur. In fact, assume the contrary. Then, reasoning similarly as in the case of (38), we can prove the existence of a regular arc

$$x = \hat{x}(t), \qquad y = \hat{y}(t) \qquad (0 \leqslant t \leqslant \hat{t}), \tag{39}$$

issuing e. g. from $(0, 1)$ (resp. $(0, -1)$), situated inside K, except its end points, and such that (3_1) is also satisfied along (39). Moreover, none of the arcs (38) and (39) can contain the other, since in this case it would contain at least three points $(1, 0)$, $(-1, 0)$, $(0, 1)$ on K^*. It follows that the arcs (38) and (39) must be disjoint, since in the opposite case there would exist meet points of (38) and (39) such that in some neighbourhood of them these arcs would be locally different. This, however, cannot occur, since in view of (i) they are both locally gemetrically defined in K by (31). On the other hand, arc (38), which connects $(1, 0)$ and $(-1, 0)$ in K, and arc (39), which connects $(0, 1)$ with one of the points $(1, 0)$, $(-1, 0)$, $(0, -1)$ in K, evidently always have common points, which yields a contradiction. In this way the third case is excluded and thus only first and second cases are possible.

In the same way we prove the desired properties of the arc

$$x = x_2(t), \quad y = y_2(t) \quad (0 \leqslant t \leqslant t_2), \tag{40}$$

which corresponds to system (31)-(32_2). So there remains to show that the arcs (38) and (40) are disjoint, in particular that each of the cases

$$x_1(t_1) = x_2(t_2) = 0, \quad y_1(t_1) = y_2(t_2) = 1,$$

$$x_1(t_1) = x_2(t_2) = 0, \quad y_1(t_1) = y_2(t_2) = -1,$$

is impossible. We show this by a reasoning similar to that concerning arcs (38) and (39). Consequently, arcs (38) and (40) have all properties required by Lemma 4 and thus the first part of this Lemma is proved. As announced above, we omit the proof of the second part of Lemma 4.

6. We shall now investigate meet points of the arcs defined in the parts 1° and 2° of Lemma 4, which we will call shortly arcs of first and second type respectively.

Considering first the arcs of the second type we see that each of the points (5_2) is connected by some of them with one of the remaining points (5_2). Further, considering the arcs of the first type we see that the following cases are possible (comp. Fig. 2):

1. One of the arcs connects $(1, 0)$ and $(0, 1)$ while the second of them connects $(-1, 0)$ and $(0, -1)$. Then the first one disconnects K between $(2^{-1/2}, 2^{-1/2})$ and the remaining three points from (5_2); consequently this arc must have meet points with that arc of the second type which connects $(2^{-1/2}, 2^{-1/2})$ with one of the remaining points (5_2). The second arc disconnects K analogously between the antipodal point $(-2^{-1/2}, -2^{-1/2})$ and the remaining ones from (5_2) and, consequently, it must have meet points with that arc of the second type which connects $(-2^{-1/2}, -2^{-1/2})$ with one of the remaining points (5_2). Thus, in any case, there exist at least two meet points of the arcs of the first type with the arcs of the second type.

2. One of the arcs connects $(1, 0)$ and $(0, -1)$ while the second of them connects $(-1, 0)$ and $(0, 1)$. Reasoning in an analogous way as in the first case we show again that there exist at least two meet points of the arcs of the first type with the arcs of the second type.

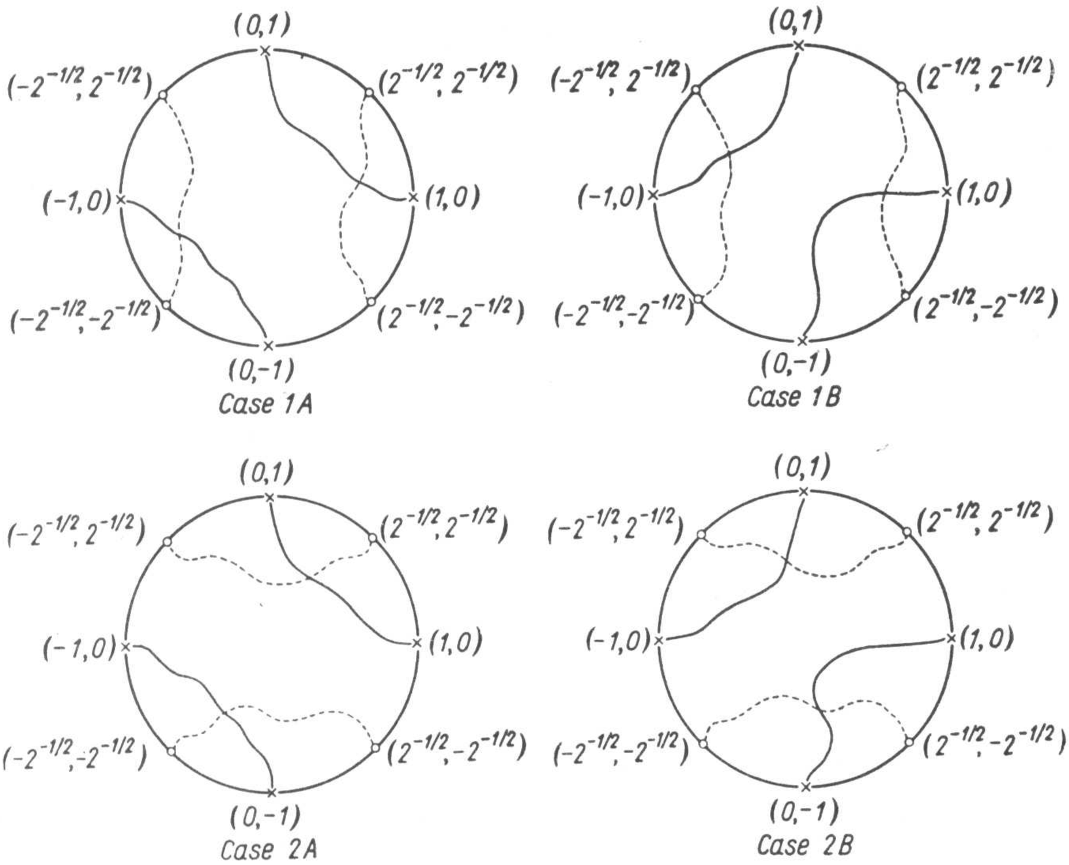

Fig. 2

Notice now that since the Cauchy-Riemann equations (3_1) and (3_2) are satisfied along the arcs of the first and of the second type, the meet points of these arcs are conformality points of mapping (1). Thus we have

THEOREM 2. *Any regular* ([4]) *J-mapping* (1) *such that the set* P *is empty, has at least two conformality points inside* K.

From Theorems 1 and 2 follows immediately

THEOREM 3. *Any regular J-mapping* (1) *has at least one conformality point in the closed unit disc* K.

7. The obtained result enlightens the hypothesis of Loewner according to which any one-to-one mapping of the unit disc K onto itself, belonging to the class C^1, preserving the boundary and satisfying inside of K relations (2), has at least one conformality point in the interior of K.

([4]) The regular mapping was defined on p. 85.

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