COLLOQUIUM MATHEMATICUM

VOL. XLVIII 1984 FASC. 1

ON RECURRENT SPACE WITH RESPECT TO METRIC SEMI-SYMMETRIC CONNECTION

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1. Introduction. Let M be an n-dimensional Riemannian manifold admitting a metric semi-symmetric connection. The purpose of this paper is to define the recurrent space with respect to metric semi-symmetric connection $\mathring{\nabla}$ and to study its recurrent properties with respect to metric semi-symmetric connection. If the Ricci tensor with respect to $\mathring{\nabla}$ is zero, then a sufficient condition for a semi-symmetrically recurrent space is obtained in terms of the vector field associated with the conformal transformation and the conformal curvature tensor. This shows that either the space is conformally flat or the vector field associated with the smooth function p is null whenever the space is Riemannian recurrent. Finally, we consider the conformal transformation and prove some properties related with it.

2. Preliminaries. Let M be the Riemannian manifold with metric tensor g_{ji} and a metric semi-symmetric connection [3]. Then Γ_{ji}^h is given by

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} + \delta_j^h p_i - g_{ji} p^h, \tag{2.1}$$

where $\left\{ {h \atop ji} \right\}$ denotes the Christoffel symbol, p_i is a covector, and $p^h = g^{hi} p_i$. The curvature tensors $\mathring{K}_{kji}{}^h$ of Γ_{ji}^h and $K_{kji}{}^h$ of $\left\{ {h \atop ji} \right\}$ are related by

$$\mathring{K}_{kji}{}^h = K_{kji}{}^h - P_{ji} \delta_k^h + P_{ki} \delta_j^h - g_{ji} A_k{}^h + g_{ki} A_j{}^h, \tag{2.2}$$

where P_{ji} is given by

$$P_{ji} = \nabla_j p_i - p_j p_i + \tfrac{1}{2} g_{ji} p_h p^h. \tag{2.3}$$

∇ denotes the covariant differentiation with respect to $\left\{ {h \atop ji} \right\}$ and $A_j{}^h = g^{hi} P_{ji}$. The concepts related with metric semi-symmetric connection are denoted by the letters with the circle above them. Let $\mathring{K}_{ji}$ be the Ricci tensor with respect to $\mathring{\nabla}$.

Since the connection Γ^h_{ji} is linear, we define the recurrency with respect to $\mathring{\nabla}$ as the space in which the curvature tensor of metric semi-symmetric connection satisfies the equality

$$\mathring{\nabla}_l \mathring{K}_{kji}{}^h = \mathring{k}_l \mathring{K}_{kji}{}^h, \tag{2.4}$$

where $\mathring{k}_l$ is the non-zero vector field of M. Such a space will be called a *semi-symmetrically recurrent space* (shortly, an SSR_n *space*). For $n > 3$, the Weyl conformal curvature tensor of M is given by

$$C_{kjih} = K_{kjih} - \frac{1}{n-2}(g_{kh}K_{ji} - g_{jh}K_{ki} + K_{kh}g_{ji} - K_{jh}g_{ki}) + \\ + \frac{K}{(n-1)(n-2)}(g_{kh}g_{ji} - g_{jh}g_{ki}),$$

where K_{ji} and K are the Ricci tensor and the scalar curvature of M, respectively.

3. Recurrency relative to metric semi-symmetric connection. Throughout this section and the next one we assume that the dimension of M is greater than 3. We now prove the main theorems.

THEOREM 3.1. *Let (M, g) be an SSR_n-space with vanishing Ricci tensor. Then the conformal curvature tensor satisfies*

$$p^h C_{kjil} + p_k C_{lji}{}^h + p_j C_{kli}{}^h + p_i C_{kjl}{}^h = 0 \tag{3.1}$$

whenever (M, g) is Riemannian recurrent.

Proof. The covariant differentiation of (2.2) with respect to $\mathring{\nabla}$ yields

$$\mathring{\nabla}_l \mathring{K}_{kji}{}^h = \mathring{\nabla}_l K_{kji}{}^h - \delta_k{}^h \mathring{\nabla}_l P_{ji} + \delta_j{}^h \nabla_l P_{ki} - g_{ji} \mathring{\nabla}_l P_k{}^h + g_{ki} \mathring{\nabla}_l P_j{}^h, \tag{3.2}$$

where P_{ij} is given by (2.3). Using (2.4) and (2.1) in (3.2), we get

$$\mathring{k}_l \mathring{K}_{kji}{}^h = \nabla_l K_{kji}{}^h + \delta_l{}^h p_t \mathring{K}_{kji}{}^t - p^h \mathring{K}_{kjil} - \\ - p_k \mathring{K}_{lji}{}^h + g_{lk} p^t \mathring{K}_{tji}{}^h - p_j \mathring{K}_{kli}{}^h + g_{lj} p^t \mathring{K}_{kti}{}^h - p_i \mathring{K}_{kjl}{}^h + g_{li} p^t \mathring{K}_{kjt}{}^h - \delta_k{}^h \mathring{\nabla}_l P_{ji} + \\ + \delta_j{}^h \mathring{\nabla}_l P_{ki} - g_{ji} \mathring{\nabla}_l P_k{}^h + g_{ki} \mathring{\nabla}_l P_j{}^h. \tag{3.3}$$

Substituting for $\mathring{K}_{kji}{}^h$ the right-hand side of (2.2), we obtain

$$\mathring{k}_l K_{kji}{}^h = \nabla_l K_{kji}{}^h + \delta_l{}^h p_t \mathring{K}_{kji}{}^t - p^h \mathring{K}_{kjil} - \\ - p_k \mathring{K}_{lji}{}^h + g_{lk} p^t \mathring{K}_{tji}{}^h - p_j \mathring{K}_{kli}{}^h + g_{lj} p^t \mathring{K}_{kti}{}^h - p_i \mathring{K}_{kjl}{}^h + \\ + g_{li} p^t \mathring{K}_{kjt}{}^h - \delta_k{}^h(\mathring{\nabla}_l P_{ji} - \mathring{k}_l P_{ji}) + \delta_j{}^h(\mathring{\nabla}_l P_{ki} - \mathring{k}_l P_{ki}) - \\ - g_{ji}(\mathring{\nabla}_l P_k{}^h - \mathring{k}_l P_k{}^h) + g_{ki}(\mathring{\nabla}_l P_j{}^h - \mathring{k}_l P_j{}^h). \tag{3.4}$$

If (M, g) is Riemannian recurrent, then

$$\text{(3.5)}\quad \delta_l^h p_t \mathring{K}_{kji}{}^t - p^h \mathring{K}_{kjil} - p_k \mathring{K}_{lji}{}^h + g_{lk} p^t \mathring{K}_{tji}{}^h - p_j \mathring{K}_{kli}{}^h + + g_{lj} p^t \mathring{K}_{kti}{}^h - p_i \mathring{K}_{kjl}{}^h + g_{li} p^t \mathring{K}_{kjt}{}^h - \delta_k^h(\mathring{\nabla}_l P_{ji} - \mathring{k}_l P_{ji}) + + \delta_j^h(\mathring{\nabla}_l P_{ki} - \mathring{k}_l P_{ki}) - g_{ji}(\mathring{\nabla}_l P_k{}^h - \mathring{k}_l P_k^h) + g_{ki}(\mathring{\nabla}_l P_j{}^h - \mathring{k}_l P_j{}^h) = 0.$$

Contracting (3.5) with respect to h and k and taking into account the vanishing of the Ricci tensor, we get

$$\text{(3.6)}\qquad (n-2)(\mathring{\nabla}_l P_{ji} - \mathring{k}_l P_{ji}) = -g_{ji}(\mathring{\nabla}_l P - \mathring{k}_l P),$$

where $P = g^{ji} P_{ji}$. Transvection of (3.6) with g^{ji} yields

$$\text{(3.7)}\qquad (n-1)(\mathring{\nabla}_l P - \mathring{k}_l P) = 0.$$

From (3.6) and (3.7) we have

$$\text{(3.8)}\qquad (n-2)(\mathring{\nabla}_l P_{ji} - \mathring{k}_l P_{ji}) = 0.$$

Since $n > 2$, from (3.8) we obtain

$$\text{(3.9)}\qquad \mathring{\nabla}_l P_{ji} - \mathring{k}_l P_{ji} = 0.$$

Substitution of (3.9) in (3.5) gives

$$\text{(3.10)}\quad \delta_l{}^h p_t \mathring{K}_{kji}{}^t - p^h \mathring{K}_{kjil} - p_k \mathring{K}_{lji}{}^h + g_{lk} p^t \mathring{K}_{tji}{}^h - - p_j \mathring{K}_{kli}{}^h + g_{lj} p^t \mathring{K}_{kti}{}^h - p_i \mathring{K}_{kjl}{}^h + g_{li} p^t \mathring{K}_{kjt}{}^h = 0.$$

Now, contracting (3.10) with respect to h and l and using the equality $\mathring{K}_{kji}{}^h = C_{kji}{}^h$, we get $(n-1) p_t C_{kji}{}^t = 0$. Since $n > 1$, we have

$$\text{(3.11)}\qquad p_t C_{kji}{}^t = 0.$$

Substituting (3.11) into (3.10) and transvecting the resulting equation with g_{hm}, we get (3.1).

THEOREM 3.2. *Let (M, g) be an SSR_n space with vanishing Ricci tensor. Then either the vector field p_i is null or the space (M, g) is conformally flat $(n > 3)$ whenever (M, g) is a Riemannian recurrent space.*

Proof. Since $\mathring{K}_{ij} = 0$, we have (see [1]) $\mathring{K}_{kji}{}^h = C_{kji}{}^h$. Transvecting (3.1) with p_h and using (3.11), we find $(p^h p_h) C_{kjil} = 0$, which completes the proof.

Note. We have assumed that (M, g) is Riemannian recurrent with recurrence vector field $\mathring{k}_l$, that is $k_l = \mathring{k}_l$.

4. Conformal transformation. Suppose that a Riemannian manifold with metric tensor g admits a metric semi-symmetric connection whose Ricci tensor vanishes. Then we can see ([4], [3]) that there exists a local function p

such that $\pi = dp$. Denote by p_i the local component of the 1-form π. We thus consider the conformal change of the metric

$$g^* = e^{2p} g. \tag{4.1}$$

Setting $\mathring{K}_{kjih} = g_{hl} \mathring{K}_{kji}{}^{l}$ we have $\mathring{K}_{kjhi} = -\mathring{K}_{kjih}$ and $\mathring{K}_{jkhi} = -\mathring{K}_{kjih}$. Since π is a closed 1-form on M, the curvature tensor of metric semi-symmetric connection satisfies the Bianchi first identity and, therefore, we get (see [4]) $\mathring{K}_{ihkj} = \mathring{K}_{kjih}$. Any tensor with respect to g^* and the corresponding tensor with respect to g will be denoted by the same letter with asterisk. If the smooth function p induces the metric semi-symmetric connection $\mathring{\nabla}$ and the conformal change of the connection $\overset{*}{\nabla}$, then

$$\overset{*}{\Gamma}{}^h_{ji} = \left\{ {h \atop ji} \right\} + \delta^h_j p_i + \delta^h_i p_j - g_{ji} p^h, \tag{4.2}$$

where $\overset{*}{\Gamma}{}^h_{ji}$ are the components of $\overset{*}{\nabla}$. From (2.1) and (4.2) we get

$$\mathring{\Gamma}{}^h_{ji} = \overset{*}{\Gamma}{}^h_{ji} - \delta^h_i p_j.$$

We find that (see [3]) $\mathring{K}_{kjih} = \overset{*}{K}_{kjih}$, where $\overset{*}{K}_{kjih}$ is the curvature tensor with respect to g^*. For easy reference we quote the following theorem due to Yano [5].

THEOREM A. *In order that a Riemannian manifold admit a metric semi-symmetric connection whose curvature vanishes it is necessary and sufficient that the Riemannian manifold be conformally flat* ($n > 3$).

THEOREM 4.1. *If* (M, g) *is an* SSR_n *space with vanishing Ricci tensor, then either the recurrence vector field of* (M, g) *with respect to the* SSR_n *space is twice the vector field associated with the conformal transformation or the space* (M, g) *is conformally flat* ($n > 3$).

Proof. Let $\mathring{k}_l$ be a non-zero recurrence vector field with respect to an SSR_n space. We have

$$\mathring{\nabla}_l \mathring{K}_{kjih} + \mathring{\nabla}_k \mathring{K}_{jlih} + \mathring{\nabla}_j \mathring{K}_{lkih} = \mathring{k}_l \mathring{K}_{kjih} + \mathring{k}_k \mathring{K}_{jlih} + \mathring{k}_j \mathring{K}_{lkih}. \tag{4.3}$$

On the other hand, we see [4] that

$$\mathring{\nabla}_l K_{kji}{}^h + \mathring{\nabla}_k \mathring{K}_{jli}{}^h + \mathring{\nabla}_j \mathring{K}_{lki}{}^h = 2(p_l \mathring{K}_{kji}{}^h + p_k \mathring{K}_{jli}{}^h + p_j \mathring{K}_{lki}{}^h). \tag{4.4}$$

Comparing (4.3) and (4.4) we find that

$$(\mathring{k}_l - 2p_l) \mathring{K}_{kjih} + (\mathring{k}_k - 2p_k) \mathring{K}_{jlih} + (\mathring{k}_j - 2p_j) K_{lkih} = 0. \tag{4.5}$$

Transvecting (4.5) with g^{lh} and using $\mathring{K}_{ji} = 0$, we have

$$(\mathring{k}_l - 2p_l) \mathring{K}_{kji}{}^l = 0. \tag{4.6}$$

Multiplying (4.5) by $\mathring{k}^l - 2p^l$ and using (4.6), we get finally

$$(\mathring{k}_l - 2p_l)(\mathring{k}^l - 2p^l) \mathring{K}_{kjih} = 0. \tag{4.7}$$

Thus from Theorem A and (4.7) we infer that either the space (M, g) is conformally flat or

$$\mathring{k}_l = 2\frac{\partial p}{\partial x^l}.$$

This completes the proof.

We know that if the Ricci tensor with respect to $\mathring{\nabla}$ vanishes, then there exists a Riemannian metric g^* $(= e^{2p}g)$ conformal to a given metric g. Thus Theorem 3.2 takes the following form:

THEOREM 4.2. *Let (M, g) be an SSR_n space with vanishing Ricci tensor. Then either the vector field associated with the conformal transformation is null or the space (M, g) is conformally flat whenever (M, g) is Riemannian recurrent.*

THEOREM 4.3. *Let (M, g) be an SSR_n space with vanishing Ricci tensor but not conformally flat. Then (M, g) is symmetric in the sense of Cartan whenever (M, g) is Riemannian recurrent.*

Proof. By Theorem 4.2, the function p in (4.1) is constant. Consequently, from (3.4) we obtain $\nabla_l K_{kji}{}^h = 0$; thus M is symmetric in the sense of Cartan.

THEOREM 4.4. *Let (M, g) be an SSR_n space with vanishing Ricci tensor. Then either the vector field associated with the conformal transformation is null or the space (M, g) is conformally flat whenever (M, g^*) is recurrent.*

Proof. Since $\mathring{\nabla}_l \mathring{K}_{kjih} = \mathring{k}_l \mathring{K}_{kjih}$, $\mathring{k}_l \neq 0$, taking covariant differentiation with respect to $\mathring{\nabla}$ we obtain

$$\mathring{\nabla}_l \mathring{K}_{kji}{}^h - \mathring{k}_l \mathring{K}_{kji}{}^h = 2p_l \mathring{K}_{kji}{}^h = 2p_l \mathring{K}_{kji}{}^h. \tag{4.8}$$

The theorem now follows from Theorem A.

THEOREM 4.5. *Let (M, g) be an SSR_n space with vanishing Ricci tensor and not conformally flat. Then $(M, e^{2p}g)$ is symmetric in the sense of Cartan whenever $(M, e^{2p}g)$ is recurrent.*

Proof. From Theorem 4.4 it follows that the vector field associated with the conformal transformation is null. Therefore, by Theorem 4.1 and (4.8), the theorem follows from (3.4).

5. Conformal curvature tensor with respect to metric semi-symmetric connection. Let p be a smooth function on M arising from the conformal change of a metric g^* on M, that is, let p be such that a metric g^* on M is conformally related to g by $g^* = e^{2p}g$. Let $\mathring{C}_{kjih}$ be the conformal curvature tensor of M relative to $\mathring{\nabla}$. Then we know [1] that the conformal curvature tensor $\mathring{C}$ with respect to $\mathring{\nabla}$ is equal to the Weyl conformal curvature tensor C of M:

$$\mathring{C}_{kjih} = C_{kjih}. \tag{5.1}$$

Differentiating (5.1) covariantly and using (2.1) we get

$$\nabla_l \mathring{C}_{kjih} = \nabla_l C_{kjih} - (C_{ljih} p_k + C_{klih} p_j + C_{kjlh} p_i + C_{kjil} p_h) + \tag{5.2}$$
$$+ p^t (C_{tjih} g_{lk} + C_{kjth} g_{li} + C_{ktih} g_{lj} + C_{kjit} g_{lh}).$$

Transvecting (5.2) with g^{lh} we obtain

$$\mathring{\nabla}_l \mathring{C}_{kji}{}^l = \nabla_l C_{kji}{}^l + (n-1) p_t C_{kji}{}^t.$$

We assume that $\mathring{\nabla}_l \mathring{C}_{kji}{}^l = 0$ and $\nabla_l C_{kji}{}^l = 0$; then

$$p_t C_{kji}{}^t = 0. \tag{5.3}$$

Multiplying (5.2) by p^h and using (5.3), we obtain $(p^h p_h) C_{kjil} = 0$. We thus have

THEOREM 5.1. *Let $\mathring{\nabla}_l \mathring{C}_{kji}{}^l = 0$. Then (M, g) is conformally symmetric iff the vector field associated with the conformal transformation is null or the space is conformally flat.*

REFERENCES

[1] K. Amur and S. S. Pujar, *On submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection*, Tensor 32 (1978), p. 36-38.
[2] M. C. Chaki and B. Gupta, *On conformally symmetric spaces*, Indian Journal of Mathematics 5 (1963), p. 113-122.
[3] T. Imai, *Notes on semi-symmetric metric connections*, Tensor 24 (1972), p. 293-296.
[4] — *Notes on semi-symmetric metric connection*, ibidem 27 (1973), p. 56-58.
[5] K. Yano, *On semi-symmetric metric connections*, Revue Roumaine de Mathématiques Purès et Appliquées 15 (1970), p. 1579-1586.

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Reçu par la Rédaction le 28. 3. 1979