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ON THE TOTAL CURVATURE OF SURFACES IN E^4

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It is well known that the total curvature of a surface is closely linked with the topology of the surface. The Gauss-Bonnet theorem is the best known example. One may expect that topological or differential topological data of the imbedding or immersing map can be used to obtain further interesting results of this kind. In this note we present some theorems dealing with properties of the curvature integral of a closed surface in the Euclidean 4-space in dependence on the imbedding or immersing map. Typical results are the following two corollaries.

COROLLARY A. *Let $f\colon S^2 \to E^4$ be an immersion of the sphere in E^4 and let H be the mean curvature, and I the self-intersection number in the sense of Whitney. Then*

$$\int\limits_{S^2} H^2 dS^2 \geqslant \pi(3+I).$$

(It is well known that I is the only immersion invariant up to regular homotopy.)

COROLLARY B. *Let $f\colon X^2 \to E^4$ be a differentiable regular imbedding of a closed 2-manifold and ϱ the minimal number of generators of the fundamental group $\pi_1(E^4 - f(X^2))$. Then*

$$\int\limits_{X^2} H^2 dX^2 \geqslant 4\pi\varrho.$$

(Note that Corollary B gives an improvement of the Willmore-Chen inequality for imbeddings with non-trivial knot group.)

1. Curvature functions. Let $x\colon X^2 \to E^4$ be an immersion of a two-dimensional manifold in the Euclidean 4-space. We use a local orthonormal moved frame $e_i(u)$ $(i = 1, 2, 3, 4)$ with $e_1(u)$ and $e_2(u)$ tangential, and $e_3(u)$ and $e_4(u)$ normal at $x(u)$. The local forms σ_1, σ_2, and ω_{ij} with

$\omega_{ij} = -\omega_{ji}$ $(i, j = 1, 2, 3, 4)$ are defined by

$$dx = e_1\sigma_1 + e_2\sigma_2 \quad \text{and} \quad de_i = \sum_{j=1}^{4} e_j\omega_{ji}.$$

Further, we have the first fundamental form

$$dx^2 = \sigma_1^2 + \sigma_2^2$$

and the two second fundamental forms

$$\alpha = \sigma_1\omega_{13} + \sigma_2\omega_{23} \quad \text{and} \quad \beta = \sigma_1\omega_{14} + \sigma_2\omega_{24}$$

depending on the choice of the normal vectors e_3 and e_4. The third fundamental form is the square of the differential of the Gauss mapping: We represent the elements of the Grassmannian manifold $G_{4,2}$ by decomposable bivectors. The Gauss mapping is defined by

$$g(u) = e_1(u) \wedge e_2(u)$$

which gives a differentiable mapping $g\colon X^2 \to G_{4,2}$. The differential of g is given by

$$dg = de_1 \wedge e_2 + e_1 \wedge de_2.$$

The third fundamental form is induced by g from the Riemannian metric of $G_{4,2}$ and is given by

$$dg^2 = \omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2.$$

We denote the components of α, β, and dg^2 with respect to the local base e_1, e_2 by $a = (a_{\alpha\beta})$, $b = (b_{\alpha\beta})$, and $c = (c_{\alpha\beta})$. The Gaussian curvature K, the mean curvature H, and two further curvatures k and h are given by

$$K = \det(a) + \det(b), \quad 4H^2 = [\operatorname{Tr}(a)]^2 + [\operatorname{Tr}(b)]^2,$$
$$2h = \operatorname{Tr}(c), \quad k^2 = \det(c).$$

The Lipschitz-Killing curvature which depends on the unit normal vector $e(\varphi) = \cos\varphi e_3 + \sin\varphi e_4$ is defined by

$$L(\varphi) = \det(\cos\varphi a + \sin\varphi b).$$

The integration over the normal vectors around a point $x(u)$ gives the mean absolute Lipschitz-Killing curvature

$$l(u) = \frac{1}{\pi}\int_0^{2\pi} |L(\varphi)|\, d\varphi.$$

The curvatures satisfy the following relations which are easy to prove:

$$k \leqslant h, \tag{1}$$

$$2H^2 = h+K, \tag{2}$$

$$l \leqslant k. \tag{3}$$

(1) is the well-known inequality between the arithmetic mean and the geometric mean of the eigenvalues of c. Relation (2) follows from $c = a^2+b^2$. To prove (3) we assume, by a suitable choice of e_3 and e_4, that $L(\varphi)$ takes the form

$$L(\varphi) = \mu_1 \sin^2\varphi + \mu_2 \cos^2\varphi$$

(Otsuki frame). By integration we get

$$l \leqslant \frac{1}{\pi} \int_0^{2\pi} |\mu_1 \sin^2\varphi + \mu_2 \cos^2\varphi| \, d\varphi \leqslant |\mu_1| + |\mu_2|.$$

We obtain $l \leqslant |\det(a)| + |\det(b)|$ for this frame. Now we apply the inequality

$$\det(a^2+b^2) \geqslant \big(|\det(a)| + |\det(b)|\big)^2 \tag{4}$$

which holds for arbitrary quadratic matrices a and b. (By multiplication with suitable unimodular matrices on both sides we can assume that a and b take the diagonal form. (4) reduces to the inequality between the arithmetic and geometric means of the diagonal elements.) The left-hand term of (4) is $\det(c) = k^2$.

Remark. As far as we know the curvature k has been considered by Leichtweiss at first, who has shown that $k = 0$ is characteristic for torses (cf. [7]).

2. Gauss mapping and total curvature. In this section we prove a theorem on closed orientable immersed surfaces. It is well known that the Whitney self-intersection number which measures the algebraic number of self-intersections is the only immersion invariant up to regular homotopy. The twice of the self-intersection number is also given by the Euler number of the normal bundle or by the degree of the spherical map from the unit tangential bundle to the unit 3-sphere (cf. [6]). It follows from the Gauss-Bonnet theorem for the normal bundle that we can write the self-intersection number as the integral

$$I = \frac{1}{4\pi} \int_{X^2} K' \, dX^2,$$

where K' is defined by $d\omega_{34} = K'\sigma_1 \wedge \sigma_2$. But we make no use of the formula here. Our aim is the proof of the following theorem:

THEOREM 1. *Let $f\colon X^2 \to E^4$ be an immersion of an orientable closed two-dimensional manifold X^2 with genus p and self-intersection number I. Then the total curvature satisfies the inequality*

$$\int_{X^2} k\,dX^2 \geqslant 2\pi(I + |p-1|).$$

Remark. Of course, the result above is not the best possible. Our interest in this result is stimulated by the following observation. The total Lipschitz-Killing curvature $\int l\,dX^2$ which is often considered in the literature (see, e.g., [3]) and also in the next section is not related to the self-intersection number. There are immersions with

$$\int l\,dX^2 \leqslant 4\pi + \varepsilon$$

for every given self-intersection number and arbitrary $\varepsilon > 0$.

Our proof of Theorem 1 is based on a result of Chern and Spanier about the homology of the Gauss mapping. In their paper [2] Chern and Spanier have given a homeomorphism of $G_{4,2}$ on $S^2 \times S^2$ which is, in fact, a homothety of Riemannian manifolds. Indeed, let

$$G_{4,2} = O(4)/SO(2) \times O(2)$$

be the Grassmannian manifold of oriented 2-planes in E^4 through the origin. As in Section 1, we consider $G_{4,2}$ as a submanifold in the Euclidean space of bivectors of E^4. The equivariant imbedding $\iota\colon G_{4,2} \to E^6$ is defined by $\iota(g) = e_1 \wedge e_2$, where g is the plane spanned by the orthogonal unit vectors e_1 and e_2. We use a fixed orthogonal base f_i in E^4 and write

$$\iota(g) = \sum_{1 \leqslant i < j \leqslant 4} a_{ij} f_i \wedge f_j.$$

The image $\iota(G_{4,2})$ is defined by the two relations $\iota(g)^2 = 1$ and $\iota(g) \wedge \iota(g) = 0$ or written in coordinates as follows:

$$\sum_{i<j} a_{ij}^2 = 1, \tag{5}$$

$$a_{12}a_{34} + a_{23}a_{14} + a_{13}a_{24} = 0. \tag{6}$$

After the orthogonal coordinate transformation

$$\begin{aligned}
\sqrt{2}x_1 &= a_{12} + a_{14}, & \sqrt{2}y_1 &= a_{12} - a_{14},\\
\sqrt{2}x_2 &= a_{13} + a_{24}, & \sqrt{2}y_2 &= a_{13} - a_{24},\\
\sqrt{2}x_3 &= a_{23} + a_{34}, & \sqrt{2}y_3 &= a_{23} - a_{34},
\end{aligned}$$

we obtain, as defining relations for $G_{4,2}^{\cdot}$ in E^6,

$$x_1^2+x_2^2+x_3^2 = 1/2 \quad \text{and} \quad y_1^2+y_2^2+y_3^2 = 1/2.$$

Therefore, $G_{4,2}$ is isometric to the Riemannian Cartesian product of the 2-sphere S_r^2 ($r^2 = 1/2$) with itself.

Let $x\colon X^2 \to E^4$ be the given immersion with the corresponding Gauss mapping $g\colon X^2 \to G_{4,2}$. The induced homomorphism of the homology groups

$$g_*\colon H_2(X^2) \to H_2(G_{4,2}) = Z \times Z$$

is calculated in [2] as

$$2g_*[X^2] = (\chi_t - \chi_n, \chi_t + \chi_n),$$

where $[X^2]$ denotes the fundamental class of X^2, and χ_n (respectively, χ_t) stands for the Euler number of the normal (respectively, tangential) bundle of X^2. In the following integrals we use absolute forms (see, e.g., [8], Chapter III). If we denote by p_1 (respectively, p_2) the first (respectively, second) projection of $G_{4,2}$ on S_r^2, then we obtain

$$\int_{X^2} k dX^2 \geqslant \int_{X^2} |(p_1 \circ g)^* dS_r^2|.$$

(We can interpret kdX^2 as a volume element of $g(X^2)$ and use the inequality $kdX^2 \geqslant |(p_1 \circ g)^* dS_r^2|$.) The second integral is estimated by

$$\int_{X^2} |(p_1 \circ g)^* dS_r^2| \geqslant \left| \int_{X^2} (p_1 \circ g)^* dS_r^2 \right| = |\deg(p_1 \circ g)| \int_{S_r^2} dS_r^2 = |\chi_t - \chi_n| \pi$$

and, analogously,

$$\int_{X^2} k dX^2 \geqslant |\chi_t + \chi_n| \pi.$$

These inequalities give together

$$\int_{X^2} k dX^2 \geqslant \pi(|\chi_t| + |\chi_n|).$$

Theorem 1 follows from $\chi_t = \chi(X^2) = 2(p-1)$ and $\chi_n = 2I$. Corollary A in the introduction follows at once from (1), (2), and the Gauss-Bonnet theorem.

3. Total curvature of knotted surfaces. In this section we consider only closed surfaces without self-intersections. Let $X^2 \to E^4$ be a closed surface. Let $\pi_1(E^4 - X^2)$ be the corresponding knot group and ϱ the minimal number of generators. We call ϱ the *knot number* of X^2.

THEOREM 2. *Let $X^2 \subset E^4$ be a closed surface with Euler characteristic number χ and knot number ϱ. Then the total absolute Lipschitz-Killing curvature satisfies the inequality*

$$\int_{X^2} l dX^2 \geqslant 2\pi(4\varrho - \chi). \tag{7}$$

Proof. We start with the following simple lemma:

LEMMA 1. *Let f be the restriction of a linear function h on X^2 which has only non-degenerate critical points on X^2. Then the number $\nu_0(f)$ of local minima satisfies $\nu_0(f) \geqslant \varrho$.*

Without loss of generality we can assume that f takes different values at the critical points p_i $(i = 0, 1, \ldots, k)$ written in the order induced from f. Let a_j be real numbers with

$$a_0 < f(p_0) < \ldots < f(p_k) < a_{k+1}.$$

By the van Kampen theorem, for the fundamental groups of the spaces $H_j = \{p; h(p) \leqslant a_j, p \notin X^2\}$ we obtain:

$\pi_1(H_{j+1}) \approx \pi_1(H_j) +$ one generator if p_j is a local minimum;

$\pi_1(H_{j+1}) \approx \pi_1(H_j) +$ one relation if p_j is a saddle point;

$\pi_1(H_{j+1}) \approx \pi_1(H_j)$ if p_j is a local maximum.

(See [5] for a more precise description of the knot group.) The lemma follows from these relations by induction on j.

We denote by ν_1 (respectively, ν_2) the number of saddle points (respectively, local maxima). Since $\nu_2(f) = \nu_0(-f)$, we have $\nu_2(f) \geqslant \varrho$, and the Morse equality gives $\nu_0 - \nu_1 + \nu_2 = \chi$. For the total number of critical points of f we obtain

$$\nu(f) \geqslant 4\varrho - \chi.$$

Theorem 2 follows now if we calculate the curvature integral via the spherical map $s\colon N^1 \to S^3$ from the unit normal bundle on the unit 3-sphere S^3. We denote by φ_e the real function $\varphi_e(u) = \langle e, u\rangle$ defined on X^2. We use a suitable normed density Ω on N^1 with $L\Omega = s^* dS^3$ and obtain

$$\pi \int_X l dX = \int_{N^1} |L|\,\Omega = \int_{N^1} s^* dS^3 = \int_{S^3} \operatorname{card}\left(s^{-1}(e)\right) dS^3$$

(see, e.g., [3]). Without loss of generality we can assume that φ_e is a Morse function. Since $\operatorname{card}\left(s^{-1}(e)\right) = \nu(\varphi_e)$, we obtain

$$\pi \int_{X^2} l dX^2 \geqslant (4\varrho - \chi) 2\pi^2$$

and the proof is complete.

From (3) we get

$$\int kdX^2 \geqslant 2\pi(4\varrho - \chi) \tag{8}$$

and from (2) we obtain Corollary B.

Finally, we give an example to show that the bounds in (8) and, therefore, in (7) are the best under the above assumptions. Let Σ_k be the k-fold connected sum of the clover leaf knots in E^3. The knot number of this knot is $k+1$ (cf. [4]). The spinning construction of Artin [1] gives a knotted sphere in E^4 with the same knot group. We have to show that this can be represented in the knot type with the total curvature

$$\int kdX < 4\pi(2k+1)+\varepsilon$$

(ε arbitrarily small). To do this we use an open arc in the half plane E^2_+ perpendicular at the two endpoints to the y-axis. We can assume that this arc has the total curvature $\int kds = \pi(2k+1)$ and is in the C_∞-topology, the limit of a sequence of open arcs representing the knot type Σ_k. (The corresponding knot is represented by the union of the arc and the segment joining the endpoints.) By rotation of this arc around the y-axis in the (x, y, z)-space we obtain an immersed sphere in $E^3 \subset E^4$ with the total curvature $\int kdS^2 = 4\pi(2k+1)$. This immersed sphere is the C_∞-limit of a sequence of imbeddings of S^2 in E^4 coming from the above-given sequence of arcs by Artin's spinning construction. It is clear that the corresponding total curvatures converge to $4\pi(2k+1)$.

It is not difficult to extend our example, by a suitable connected sum construction, to manifolds with arbitrary genus p. So we have

THEOREM 3. *Given integers $p \geqslant 0$ and $\varrho \geqslant 1$, there exists an imbedding of an orientable closed manifold X^2 with genus p and knot number ϱ with a total absolute curvature arbitrarily near* $4\pi(2\varrho+p-1)$.

Remark. The relation between the generator number of the knot group and the critical point number of a height function holds also in the higher-dimensional case. By a method like [4], Sunday [9] has recently shown an inequality for knotted spheres of dimension n in E^{n+2}, namely $\nu(f) \geqslant 2\varrho$. But for all $n > 1$ Sunday's inequality can essentially be improved (see the author's forthcoming paper [10] including also results for knotted spheres with cyclic knot group).

REFERENCES

[1] E. Artin, *Zur Isotopie zweidimensionaler Flächen in R^4*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 4 (1925), p. 174-177.

[2] S. S. Chern and E. Spanier, *A theorem on orientable surfaces in four-dimensional space*, Commentarii Mathematici Helvetici 25 (1951), p. 205-209.

[3] D. Ferus, *Totale Absolutkrümmung in Differentialgeometrie und -topologie*, Lecture Notes in Mathematics 66, Berlin 1968.

[4] R. Fox, *On the total curvature of some tame knots*, Annals of Mathematics 52 (1950), p. 258-260.

[5] S. Kinoshita, *On the Alexander polynomials of 2-spheres in a 4-sphere*, ibidem 74 (1961), p. 518-537.

[6] R. K. Lashof and S. Smale, *On the immersions of manifolds in the euclidean space*, ibidem 68 (1958), p. 562-583.

[7] K. Leichtweiss, *Über eine Art von Krümmungsinvarianten beliebiger Untermannigfaltigkeiten des n-dimensionalen euklidischen Raumes*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 26 (1963-1964), p. 155-190.

[8] R. Sulanke and P. Wintgen, *Differentialgeometrie und Faserbündel*, Berlin 1972.

[9] D. Sunday, *The total curvature of knotted spheres*, Bulletin of the American Mathematical Society 82 (1976), p. 140-142.

[10] P. Wintgen, *Zur Integralkrümmung verknoteter Sphären*, Dissertation B, Berlin 1976 (to appear in Beiträge zur Algebra und Geometrie).

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