

*PRIMITIVE CLASSES OF ALGEBRAS
WITH UNARY AND NULLARY OPERATIONS*

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Jacobs and Schwabauer [1] gave a complete description of the lattice of primitive classes of algebras with one unary operation and proved that the lattice is countable and distributive. Consider a type Δ and the lattice \mathcal{L}_Δ of primitive classes of algebras of type Δ . In the present paper we shall prove that the lattice \mathcal{L}_Δ is countable if and only if Δ consists of a finite number of nullary and at most one unary operation (Theorem 6). We shall give a complete description of these countable lattices \mathcal{L}_Δ (Theorems 1 and 2 and the remark at the end of Section 3). In Theorem 7 we shall find all the types Δ such that the lattice \mathcal{L}_Δ is distributive or modular.

1. Preliminaries. We shall suppose the knowledge of Słomiński [4]. However, only algebras with finitary operations are considered in the present paper.

By a *type* we mean a family $\Delta = (n_j)_{j \in J}$ of natural numbers ($n_j \geq 0$). By an *algebra of type* $(n_j)_{j \in J}$ we mean a set A together with a family $(f_j)_{j \in J}$ such that, for each $j \in J$, f_j is an n_j -ary operation in A . If $n_j = 0$, then f_j is an element of A ; it is denoted by o_j .

Let I_0, I_1, \dots, I_m ($m \geq 0$) be pairwise disjoint sets. Then $((I_0, I_1, \dots, I_m))$ denotes the type $(n_j)_{j \in I_0 \cup I_1 \cup \dots \cup I_m}$, where $n_j = 0$ if $j \in I_0$, $n_j = 1$ if $j \in I_1, \dots$, $n_j = m$ if $j \in I_m$.

If k_0, k_1, \dots, k_m ($m \geq 0$) are natural numbers, then $((k_0, k_1, \dots, k_m))$ denotes the type $((I_0, I_1, \dots, I_m))$, where the sets I_0, I_1, \dots, I_m are suitably chosen so that $\text{Card } I_0 = k_0$, $\text{Card } I_1 = k_1, \dots, \text{Card } I_m = k_m$. (It is the type of algebras with k_0 nullary, k_1 unary, \dots , and k_m m -ary operations.) It is clear which type is denoted by $((I, k_1, \dots, k_m))$.

Let us fix an infinite countable set and denote it by X ; its elements are called *variables*. For each type Δ let us fix an absolutely free algebra of type Δ freely generated by X , and denote it by W_Δ . The elements of W_Δ are called Δ -*terms*. The ordered pairs $\langle a, b \rangle$ of Δ -terms are called Δ -*equations*. An equation $\langle a, b \rangle$ is called *trivial* if $a = b$. The set of all

trivial Δ -equations is denoted by Triv_Δ . By a Δ -theory we mean an arbitrary set of Δ -equations, i.e. a subset of $W_\Delta \times W_\Delta$, i.e. a binary relation in W_Δ . If E is a theory, then E^{-1} denotes the set of all $\langle a, b \rangle$ such that $\langle b, a \rangle \in E$.

If E_1 and E_2 are two theories, then we shall write $E_1 \vdash E_2$ if each model of E_1 is a model of E_2 . Theories E_1 and E_2 are called *equivalent* if $E_1 \vdash E_2$ and $E_2 \vdash E_1$. If α is an equation and E a theory, then we shall write $\alpha \vdash E$ if $\{\alpha\} \vdash E$, etc.

A congruence relation E of W_Δ is called *fully invariant* if $\langle a, b \rangle \in E$ implies $\langle \varphi(a), \varphi(b) \rangle \in E$ for all endomorphisms φ of W_Δ .

Let E be a theory. Then $\text{Cn} E$ denotes the set of all the equations α such that $E \vdash \alpha$. It is just the least fully invariant congruence relation of W_Δ containing the theory E .

It is well-known that primitive classes of algebras of type Δ are in a one-to-one correspondence with the fully invariant congruence relations of W_Δ .

The lattice of all primitive classes of algebras of type Δ is denoted by \mathcal{L}_Δ and is defined in this way: its elements are just the primitive classes of algebras of type Δ , and the partial ordering of the lattice coincides with the set-theoretical inclusion. Of course, this definition is not correct: it is meaningless to speak about a set of proper classes. This incorrectness could be removed if we defined \mathcal{L}_Δ as the dual of the lattice of all fully invariant congruence relations of W_Δ . However, it will be convenient to keep the definition given above.

2. Lattices $\mathcal{L}_{((I))}$. Consider the type $((I))$, where I is a given set; this type consists only of nullary operations. Let us denote by O the set of all nullary operations o_i ($i \in I$) of the algebra $W_{((I))}$. We have $W_{((I))} = X \cup O$. Lowig [1], Satz 2.6, has proved that:

(a) Each fully invariant congruence relation of $W_{((I))}$ is equal either to $W_{((I))} \times W_{((I))}$ (i.e. to the greatest congruence relation) or to $\text{Triv}_{((I))} \cup \eta$, where η is an equivalence relation over O .

(b) If η is an equivalence relation over O , then $\text{Triv}_{((I))} \cup \eta$ is a fully invariant congruence relation of $W_{((I))}$.

Using this we can easily describe the lattice $\mathcal{L}_{((I))}$:

THEOREM 1. *Let I be a given set. Denote by K_I the dual of the equivalence lattice of the set I (so that $\eta_1 \leq \eta_2$ in K_I if and only if $\eta_1 \supseteq \eta_2$). Denote by L_I the lattice that results from K_I by adding a new element which we declare to be the least element of L_I . Then $\mathcal{L}_{((I))}$ is isomorphic to L_I .*

COROLLARY 1. *If I is finite, then $\mathcal{L}_{((I))}$ is finite. If I is infinite, then $\mathcal{L}_{((I))}$ is uncountable. If I is countably infinite, then $\mathcal{L}_{((I))}$ has exactly 2^{\aleph_0} elements.*

The proof is easy.

From Theorem 1 we easily get: $\mathcal{L}_{((0))}$ and $\mathcal{L}_{((1))}$ are both isomorphic to the lattice in Fig. 1; $\mathcal{L}_{((2))}$ is isomorphic to the lattice in Fig. 2; $\mathcal{L}_{((3))}$ is isomorphic to the lattice in Fig. 3; if $\text{Card } I \geq 4$, then $\mathcal{L}_{((I))}$ contains a sublattice which is isomorphic to the lattice in Fig. 4.



Fig. 1



Fig. 2

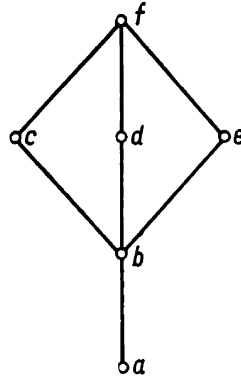


Fig. 3

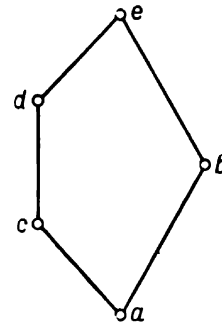


Fig. 4

COROLLARY 2. *If $k \leq 2$, then the lattice $\mathcal{L}_{((k))}$ is distributive. The lattice $\mathcal{L}_{((3))}$ is modular but not distributive. If $\text{Card } I \geq 4$, then the lattice $\mathcal{L}_{((I))}$ is not modular.*

3. Lattices $\mathcal{L}_{((I,1))}$. Let I be a given set. In this section we shall consider only algebras of type $((I, 1))$. All the equations under consideration are $((I, 1))$ -equations; similarly for theories etc. Put $W = W_{((I,1))}$, $\text{Triv} = \text{Triv}_{((I,1))}$ etc.

If A is an algebra (of the considered type) and $a \in A$, then the result of the only unary operation of A applied to a is denoted by a' . Let us define a^n inductively for all natural numbers n : $a^0 = a$; $a^{n+1} = (a^n)'$.

The set of all nullary operations o_i ($i \in I$) of the algebra W is denoted by O . Put $Y = X \cup O$. Each term can be expressed in the form z^n , where $z \in Y$, in exactly one way. If $z \in O$ ($z \in X$, resp.), then it is called a *constant term* (a *variable term*, resp.). An equation is called *constant* (*variable*, resp.) if on both its sides there are constant (variable, resp.) terms. A theory is called *constant* (*variable*, resp.) if it contains only constant (variable, resp.) equations. The set of all constant (variable, resp.) equations is denoted by C (V , resp.).

Let $i, j \in I$. The set of all the equations $\langle o_i^n, o_j^m \rangle$, where $n, m \geq 0$ are arbitrary, is denoted by $C^{i,j}$. If E is a theory, then put $E^{i,j} = E \cap C^{i,j}$; put $E^{\bar{i},\bar{j}} = E^{i,j} \cup E^{i,i} \cup E^{j,j}$. A theory E is called an $[i, j]$ -theory if $E = E^{\bar{i},\bar{j}}$.

An $[i, i]$ -theory E is called a *reduced $[i, i]$ -theory* if $E \subseteq \{\langle o_i^n, o_i^{n+d} \rangle\}$, where $d > 0$.

Let $i \neq j$. An $[i, j]$ -theory E is called a *reduced $[i, j]$ -theory* if one of the following four conditions is satisfied:

- (1) $E \subseteq \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+e} \rangle\}$, where $d, e > 0$.
- (2) $E = \{\langle o_i^k, o_j^l \rangle\}$.
- (3) $E = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle, \langle o_i^{n-c}, o_j^{m-c} \rangle\}$, where $d, c > 0$.
- (4) $E = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle, \langle o_i^n, o_j^{m+c} \rangle\}$, where $d > 0$, $0 \leq c < d$.

A constant theory E is called a *constant-reduced theory* if the following three conditions are satisfied:

- (5) If $i, j \in I$, then $E^{i,j}$ is a reduced $[i, j]$ -theory.
- (6) Let $i \neq j$. If $E^{i,j}$ satisfies (1), then $E^{j,i}$ is empty; if it satisfies (2), then $E^{j,i} = \{\langle o_j^l, o_i^k \rangle\}$; if it satisfies (3), then $E^{j,i} = \{\langle o_j^{m-c}, o_i^{n-c} \rangle\}$; if it satisfies (4), then $E^{j,i} = \{\langle o_j^m, o_i^{n+d-c} \rangle\}$.
- (7) Let i, j and h be three pairwise different elements of I and let E contain the equations $\langle o_i^k, o_j^l \rangle$ and $\langle o_j^p, o_h^q \rangle$.
 If $l < p$, then E contains the equation $\langle o_i^{k+p-l}, o_h^q \rangle$.
 If $l = p$, then there exists a $c \geq 0$ such that E contains the equation $\langle o_i^{k-c}, o_h^{q-c} \rangle$.
 Let $l > p$.
 (a) If E contains $\langle o_h^n, o_h^{n+d} \rangle$ and $q + l - p > n + d$, then E contains $\langle o_i^k, o_h^{q+l-p-d} \rangle$.
 (b) If E contains $\langle o_h^n, o_h^{n+d} \rangle$ and $q + l - p = n + d$, then there exists a $c \geq 0$ such that E contains $\langle o_i^{k-c}, o_h^{n-c} \rangle$.
 (c) In all other cases E contains $\langle o_i^k, o_h^{q+l-p} \rangle$.

Let us fix two variables x and y ($x \neq y$).

A variable theory E is called a *variable-reduced theory* if one of the following three conditions is satisfied:

- (8) E is empty.
- (9) $E = \{\langle x^n, x^{n+d} \rangle\}$, where $d > 0$.
- (10) $E = \{\langle x^n, y^n \rangle\}$.

A theory E is called *reduced* if the following five conditions are satisfied:

- (11) $E \subseteq C \cup V$.
- (12) $E \cap C$ is a constant-reduced theory.
- (13) $E \cap V$ is a variable-reduced theory.
- (14) If $E \cap V = \{\langle x^n, x^{n+d} \rangle\}$, then for each $i \in I$ there exist m and e such that $E^{i,i} = \{\langle o_i^m, o_i^{m+e} \rangle\}$, and $m \leq n$ and e divides d .
- (15) If $E \cap V = \{\langle x^n, y^n \rangle\}$, then each $E^{i,j}$ is non-empty and for each $i \in I$ there exists an $m \leq n$ such that $E^{i,i} = \{\langle o_i^m, o_i^{m+1} \rangle\}$.

Notice that each constant-reduced theory is reduced, while a variable-reduced theory E is reduced if and only if either E or I is empty.

THEOREM 2. *The lattice $\mathcal{L}_{((I,1))}$ is isomorphic to the lattice L of all reduced theories with the partial ordering defined by $E_1 \leq E_2$ if and only if $E_1 \vdash E_2$.*

The proof will be preceded by several lemmas.

LEMMA 1. *Let $i \in I$ and $E = \{\langle o_i^n, o_i^{n+d} \rangle\}$, where $d > 0$. Let M be the set of all $\langle o_i^m, o_i^{m+e} \rangle$, where $n \leq m$, $e > 0$ and d divides e . Then $\text{Cn} E = \text{Triv} \cup M \cup M^{-1}$.*

Proof. We evidently have $E \subseteq \text{Triv} \cup M \cup M^{-1} \subseteq \text{Cn} E$. It is easy to see that $\text{Triv} \cup M \cup M^{-1}$ is a fully invariant congruence relation of W .

LEMMA 2. *Let $i, j \in I$, $i \neq j$ and let E be a reduced $[i, j]$ -theory. Let us describe the set $\text{Cn} E$ by the following conditions:*

(a) *If $E \subset \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+e} \rangle\}$, where $d, e > 0$, then $\text{Cn} E$ is described in Lemma 1.*

(b) *If $E = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+e} \rangle\}$, where $d, e > 0$, then*

$$\text{Cn} E = \text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle\} \cup \text{Cn}\{\langle o_j^m, o_j^{m+e} \rangle\}.$$

(c) *If $E = \{\langle o_i^k, o_j^l \rangle\}$, then*

$$\text{Cn} E = \text{Triv} \cup M \cup M^{-1},$$

where M is the set of all the equations $\langle o_i^{k+c}, o_j^{l+c} \rangle$ such that $c \geq 0$.

(d) *If $E = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle, \langle o_i^{n-c}, o_j^{m-c} \rangle\}$, where $d, c > 0$, then*

$$\text{Cn} E = M \cup M^{-1} \cup \text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle\},$$

where M is the set of all the equations $\langle o_i^r, o_j^s \rangle$ such that either $0 \leq r - (n - c) = s - (m - c)$ or $r \geq n$, $s \geq m$ and d divides $(r - n) - (s - m)$.

(e) *If $E = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle, \langle o_i^n, o_j^{m+c} \rangle\}$, where $d > 0$ and $0 \leq c < d$, then*

$$\text{Cn} E = M \cup M^{-1} \cup \text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle\},$$

where M is the set of all the equations $\langle o_i^r, o_j^s \rangle$ such that $r \geq n$, $s \geq m$ and d divides $(r - n) - (s - (m + c))$.

Proof. The proofs of (b) and (c) are easy and analogous to the proof of Lemma 1.

(d) Put

$$\bar{M} = M \cup M^{-1} \cup \text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle\}.$$

It is sufficient to prove that \bar{M} is a fully invariant congruence relation of W ; to prove this, it is evidently sufficient to prove that \bar{M} is

a transitive relation. Let $\langle a_1, a_2 \rangle, \langle a_2, a_3 \rangle \in \bar{M}$. If both these equations belong to

$$\text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle\},$$

then evidently $\langle a_1, a_3 \rangle \in \bar{M}$. We may confine ourselves to the case $\langle a_1, a_2 \rangle \in M$ (all the other cases are similar). We have $\langle a_1, a_2 \rangle = \langle o_i^r, o_j^s \rangle$. Let, firstly, $r \geq n$, $s \geq m$ and d divide $(r-n)-(s-m)$. The equation $\langle a_2, a_3 \rangle$ is either equal to $\langle o_j^s, o_i^{\bar{s}} \rangle$, where $\bar{s} \geq m$ and d divides $s-\bar{s}$, or it is equal to $\langle o_j^s, o_i^{\bar{r}} \rangle$, where $\bar{r} \geq n$ and d divides $(\bar{r}-n)-(s-m)$; in the first case we have $r \geq n$, $\bar{s} \geq m$ and d divides $(r-n)-(s-m)+s-\bar{s} = (r-n)-(\bar{s}-m)$, so that $\langle a_1, a_3 \rangle = \langle o_i^r, o_j^{\bar{s}} \rangle \in M \subseteq \bar{M}$; in the second case we have $r \geq n$, $\bar{r} \geq n$ and d divides $((r-n)-(s-m))-((\bar{r}-n)-(s-m)) = r-\bar{r}$, so that again

$$\langle a_1, a_3 \rangle = \langle o_i^r, o_i^{\bar{r}} \rangle \in \text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle\} \subseteq \bar{M}.$$

Let, secondly, $0 \leq r-(n-c) = s-(m-c)$. It is sufficient to suppose $r < n$ (and hence $s < m$): if we had $r \geq n$, the already considered case would take place. But then $\langle a_2, a_3 \rangle$ is either trivial or equal to $\langle o_j^s, o_i^{\bar{r}} \rangle$, so that $\langle a_1, a_3 \rangle \in \bar{M}$.

(e) Put

$$\bar{M} = M \cup M^{-1} \cup \text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle\}.$$

As above, it is sufficient to prove that if $\langle a_1, a_2 \rangle \in M$ and $\langle a_2, a_3 \rangle \in \bar{M}$, then $\langle a_1, a_3 \rangle \in \bar{M}$. We have $\langle a_1, a_2 \rangle = \langle o_i^r, o_j^s \rangle$, where $r \geq n$, $s \geq m$ and d divides $(r-n)-(s-(m+c))$. Let, firstly, $\langle a_2, a_3 \rangle = \langle o_j^s, o_i^{\bar{s}} \rangle$. Then $\bar{s} \geq m$ and d divides $s-\bar{s}$; we have $r \geq n$, $\bar{s} \geq m$ and d divides $(r-n)-(s-(m+c))+s-\bar{s} = (r-n)-(\bar{s}-(m+c))$, so that $\langle a_1, a_3 \rangle = \langle o_i^r, o_j^{\bar{s}} \rangle \in M \subseteq \bar{M}$. Secondly, let $\langle a_2, a_3 \rangle = \langle o_j^s, o_i^{\bar{r}} \rangle$. Then $\bar{r} \geq n$ and d divides $(\bar{r}-n)-(s-(m-c))$, and hence d divides $((r-n)-(s-(m-c)))-((\bar{r}-n)-(s-(m-c))) = r-\bar{r}$, so that again

$$\langle a_1, a_3 \rangle = \langle o_i^r, o_i^{\bar{r}} \rangle \in \text{Cn}\{\langle o_i^n, o_i^{n+d} \rangle\} \subseteq \bar{M}.$$

LEMMA 3. Let I be non-empty and let E be a constant-reduced theory. Then

$$\text{Cn} E = \bigcup_{i,j \in I} \text{Cn} E^{i,j}.$$

Proof. Put $M = \bigcup_{i,j \in I} \text{Cn} E^{i,j}$. It is clearly sufficient to prove that M is a fully invariant congruence relation of W . Everything except the transitivity of M is evident; as for the transitivity, it is sufficient to prove the following assertion:

- (16) If i, j and h are three pairwise different elements of I and if $\langle o_i^r, o_j^s \rangle \in M$, and $\langle o_j^s, o_h^t \rangle \in M$, then $\langle o_i^r, o_h^t \rangle \in M$.

Proof. It follows easily from Lemma 2 that $\langle o_i^r, o_j^s \rangle \in \text{Cn } \overline{E^{i,j}}$ and $\langle o_j^s, o_h^t \rangle \in \text{Cn } \overline{E^{j,h}}$. Evidently, the theories $E^{i,j}$ and $E^{j,h}$ are non-empty and hence they both have exactly one element; put $E^{i,j} = \{\langle o_i^k, o_j^l \rangle\}$ and $E^{j,h} = \{\langle o_j^p, o_h^q \rangle\}$.

Consider first the case

$$(17) \quad 0 \leq r-k = s-l \quad \text{and} \quad 0 \leq s-p = t-q.$$

If $l < p$, then by (7) we get $\langle o_i^{k+p-l}, o_h^q \rangle \in E^{i,h}$, so that

$$M \supseteq \text{Cn } E^{i,h} \ni \langle o_i^{k+p-l+s-p}, o_h^{q+s-p} \rangle = \langle o_i^r, o_h^t \rangle.$$

If $l = p$, then by (7) we infer $\langle o_i^k, o_h^q \rangle \in \text{Cn } E^{i,h}$, so that

$$M \supseteq \text{Cn } E^{i,h} \ni \langle o_i^{k+r-k}, o_h^{q+r-k} \rangle = \langle o_i^r, o_h^t \rangle.$$

If $p < l$, then by (7) we infer in all the three possible cases that $\langle o_i^k, o_h^{q+l-p} \rangle \in \text{Cn } \overline{E^{i,h}}$, so that again

$$M \supseteq \text{Cn } \overline{E^{i,h}} \ni \langle o_i^{k+s-l}, o_h^{q+l-p+s-l} \rangle = \langle o_i^r, o_h^t \rangle.$$

We have proved (16) in the case (17). From now on we may suppose that (17) does not hold. The theories $E^{i,i}$, $E^{j,j}$ and $E^{h,h}$ are evidently non-empty; we have

$$\overline{E^{i,i}} = \{\langle o_i^{n_1}, o_i^{n_1+d} \rangle, \langle o_j^{n_2}, o_j^{n_2+d} \rangle, \langle o_i^k, o_j^l \rangle\}$$

and

$$\overline{E^{j,h}} = \{\langle o_j^{n_2}, o_j^{n_2+d} \rangle, \langle o_h^{n_3}, o_h^{n_3+d} \rangle, \langle o_j^p, o_h^q \rangle\}$$

(where $d > 0$).

As (17) does not hold, we easily infer from Lemma 2 that $r \geq n_1$, $s \geq n_2$ and $t \geq n_3$. Let us prove the following assertion:

$$(18) \quad \text{There exist numbers } \bar{s} \geq n_2 \text{ and } \bar{t} \geq n_3 \text{ such that } \bar{s} < n_2 + d, \langle o_i^{n_1}, o_j^{\bar{s}} \rangle \in \text{Cn } \overline{E^{i,j}}, \langle o_j^{\bar{s}}, o_h^{\bar{t}} \rangle \in \text{Cn } \overline{E^{j,h}} \text{ and } \langle o_i^{n_1}, o_h^{\bar{t}} \rangle \in \text{Cn } \overline{E^{i,h}}.$$

The following four cases are possible:

(a) $0 < n_1 - k = n_2 - l$ and $0 < n_2 - p = n_3 - q$. We may put $\bar{s} = n_2$ and $\bar{t} = n_3$.

(b) $0 < n_1 - k = n_2 - l$, $p = n_2$ and $q = n_3 + c$, where $0 \leq c < d$. Put $\bar{s} = n_2$ and $\bar{t} = n_3 + c$. By (7) we get $\langle o_i^{n_1}, o_h^{\bar{t}} \rangle \in \overline{E^{i,h}}$. We have $\langle o_i^{n_1}, o_j^{\bar{s}} \rangle \in \text{Cn } \overline{E^{i,j}}$ and $\langle o_j^{\bar{s}}, o_h^{\bar{t}} \rangle \in \overline{E^{j,h}}$.

(c) $k = n_1$, $l = n_2 + c$ and $0 < n_2 - p = n_3 - q$, where $0 \leq c < d$. Put $\bar{s} = n_2 + c$ and $\bar{t} = n_3 + c$. By (7) we get $\langle o_i^{n_1}, o_h^{\bar{t}} \rangle \in \overline{E^{i,h}}$. We have $\langle o_i^{n_1}, o_j^{\bar{s}} \rangle \in \overline{E^{i,j}}$ and $\langle o_j^{\bar{s}}, o_h^{\bar{t}} \rangle \in \text{Cn } \overline{E^{j,h}}$.

(d) $k = n_1$, $l = n_2 + c_1$, $p = n_2$ and $q = n_3 + c_2$, where $0 \leq c_1 < d$, $0 \leq c_2 < d$. Put $\bar{s} = n_2 + c_1$, $\bar{t} = n_3 + c_1 + c_2$. By (7) we get (in all the three possible cases) $\langle o_i^{n_1}, o_h^{\bar{t}} \rangle \in \text{Cn } E^{\bar{t}, \bar{h}}$. We have $\langle o_i^{n_1}, o_j^{\bar{s}} \rangle \in E^{\bar{t}, \bar{j}}$ and $\langle o_j^{\bar{s}}, o_h^{\bar{t}} \rangle \in \text{Cn } E^{\bar{j}, \bar{h}}$.

We have proved (18).

Let us finish the proof of (16). We have $r \geq n_1$, $s \geq n_2$ and $t \geq n_3$.

There exists exactly one c_1 such that

$$(19) \quad 0 \leq c_1 < d \quad \text{and} \quad \langle o_i^{n_1}, o_i^{n_1+d} \rangle \vdash \langle o_i^{n_1+c_1}, o_i^r \rangle,$$

and exactly one c_2 such that

$$(20) \quad 0 \leq c_2 < d \quad \text{and} \quad \langle o_j^{n_2}, o_j^{n_2+d} \rangle \vdash \langle o_j^{n_2+c_2}, o_j^{s+d-c_1} \rangle.$$

We have $E^{\bar{t}, \bar{j}} \vdash \langle o_i^{r+d-c_1}, o_j^{s+d-c_1} \rangle$; by (19) and (20) we get

$$(21) \quad E^{\bar{t}, \bar{j}} \vdash \langle o_i^{n_1}, o_j^{n_2+c_2} \rangle.$$

By (18) there exist numbers $\bar{s} \geq n_2$ and $\bar{t} \geq n_3$ such that $\bar{s} < n_2 + d$ and

$$(22) \quad E^{\bar{t}, \bar{j}} \vdash \langle o_i^{n_1}, o_j^{\bar{s}} \rangle, \quad E^{\bar{j}, \bar{h}} \vdash \langle o_j^{\bar{s}}, o_h^{\bar{t}} \rangle, \quad E^{\bar{t}, \bar{h}} \vdash \langle o_i^{n_1}, o_h^{\bar{t}} \rangle.$$

By (21) and (22) we have $E^{\bar{t}, \bar{j}} \vdash \langle o_j^{n_2+c_2}, o_j^{\bar{s}} \rangle$, so that $\bar{s} = n_2 + c_2$. By this and by (22) and (20) we get

$$E^{\bar{j}, \bar{h}} \vdash \langle o_h^{\bar{t}+c_1}, o_j^{n_2+c_2+c_1} \rangle, \langle o_j^{n_2+c_2+c_1}, o_j^{s+d} \rangle, \langle o_j^{s+d}, o_j^s \rangle, \langle o_j^s, o_h^{\bar{t}} \rangle;$$

whence, by Lemma 2,

$$(23) \quad \langle o_h^{n_3}, o_h^{n_3+d} \rangle \vdash \langle o_h^{t+c_1}, o_h^t \rangle.$$

By (19), (22) and (23) we get $E^{\bar{t}, \bar{h}} \vdash \langle o_i^r, o_h^t \rangle$. Hence $\langle o_i^r, o_h^t \rangle \in \text{Cn } E^{\bar{t}, \bar{h}} \subseteq M$, q.e.d.

LEMMA 4. Let $i \in I$; let two equations $\alpha = \langle o_i^n, o_i^{n+d} \rangle$ and $\beta = \langle o_i^m, o_i^{m+e} \rangle$, where $d, e > 0$, be given. Then either $\alpha \vdash \beta$ or there exists an equation $\gamma = \langle o_i^p, o_i^q \rangle$ such that $\{\alpha, \beta\} \vdash \gamma$ and either $p = n \neq q$ and $q < n + d$ or $p < n$ and $p \neq q$.

Proof. Let c_1 be the least non-negative integer such that $m - c_1 d < n + d$. Let c_2 be the least non-negative integer such that $m + e - c_2 d < n + d$. Evidently,

$$(24) \quad \alpha \vdash \langle o_i^m, o_i^{m-c_1 d} \rangle \quad \text{and} \quad \alpha \vdash \langle o_i^{m+e}, o_i^{m+e-c_2 d} \rangle.$$

Put $\delta = \langle o_i^{m-c_1 d}, o_i^{m+e-c_2 d} \rangle$. By (24) we get $\{\alpha, \beta\} \vdash \delta$ and $\{\alpha, \delta\} \vdash \beta$. The theories $\{\alpha, \beta\}$ and $\{\alpha, \delta\}$ are thus equivalent.

If $m - c_1 d = m + e - c_2 d$, then evidently $\alpha \vdash \beta$. Let $m - c_1 d \neq m + e - c_2 d$; let us denote by k the least and by l the greatest of these two

numbers. Put $\bar{\beta} = \langle o_i^k, o_i^l \rangle$, so that the theories $\{\alpha, \beta\}$ and $\{\alpha, \bar{\beta}\}$ are equivalent.

Let $k < n$. We have $\{\alpha, \bar{\beta}\} \vdash \bar{\beta}$ and hence $\{\alpha, \beta\} \vdash \bar{\beta}$; we may put $\gamma = \bar{\beta}$.

Let $k \geq n$; put $\gamma = \langle o_i^n, o_i^{n+d-l+k} \rangle$. As $l < n+d$, we have $\bar{\beta} \vdash \langle o_i^{n+d}, o_i^{n+d-l+k} \rangle$. Hence $\{\alpha, \bar{\beta}\} \vdash \gamma$, and, consequently, $\{\alpha, \beta\} \vdash \gamma$. As $n+d > l$, we have $n+d-l+k > k \geq n$, so that $n+d-l+k \neq n$; the inequality $n+d-l+k < n+d$ is evident. Lemma 4 is proved.

Let E be an arbitrary theory. Let us construct a constant theory $T = \bigcup_{i,j \in I} T^{i,j}$ as follows: If $i \in I$, then $T^{i,i} = \{\langle o_i^n, o_i^{n+d} \rangle\}$, where n is the least number such that there exists a $d_0 > 0$ with $E \vdash \langle o_i^n, o_i^{n+d_0} \rangle$, and d is the least number such that $E \vdash \langle o_i^n, o_i^{n+d} \rangle$; if such an n does not exist, then $T^{i,i}$ is empty. If $i, j \in I$ and $i \neq j$, then $T^{i,j} = \{\langle o_i^k, o_j^l \rangle\}$, where k is the least number such that there exists an l_0 with $E \vdash \langle o_i^k, o_j^{l_0} \rangle$, and l is the least number such that $E \vdash \langle o_i^k, o_j^l \rangle$; if such a k does not exist, then $T^{i,j}$ is empty. The theory T constructed in this way is denoted by $\mathcal{R}_c(E)$.

LEMMA 5. *Let E be an arbitrary theory. Then $\mathcal{R}_c(E)$ is a constant-reduced theory.*

Proof. Put $T = \mathcal{R}_c(E)$. If $i \in I$, then $T^{i,i} = T^{i,i}$ is evidently a reduced $[i, i]$ -theory. Let $i, j \in I$, $i \neq j$. We shall prove that $T^{i,j}$ is a reduced $[i, j]$ -theory. If T contains no equations $\langle o_i^k, o_j^l \rangle$, then $T^{i,j}$ satisfies (1). Let T contain an equation $\langle o_i^k, o_j^l \rangle$. If $T^{i,j}$ contains only this one equation, then it satisfies (2); in the opposite case we evidently have $T^{i,j} = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+e} \rangle, \langle o_i^k, o_j^l \rangle\}$, where $d, e > 0$. Let us consider the latter case.

There exist numbers r and s such that $r \geq n, s \geq m$ and $E \vdash \langle o_i^r, o_j^s \rangle$. We have $E \vdash \langle o_i^r, o_i^{r+d} \rangle, E \vdash \langle o_i^{r+d}, o_j^{s+d} \rangle$ and, consequently, $E \vdash \langle o_j^s, o_j^{s+d} \rangle$. By Lemma 4 it can now be easily proved that $\langle o_j^m, o_j^{m+e} \rangle \vdash \langle o_j^s, o_j^{s+d} \rangle$. By Lemma 1, e divides d . Quite similarly, d divides e . We get $d = e$.

Evidently, there exists an $r \geq 0$ such that $k+r \geq n$ and d divides $k+r-n$. We have $\langle o_i^k, o_j^l \rangle \vdash \langle o_i^{k+r}, o_j^{l+r} \rangle$ and $\langle o_i^n, o_i^{n+d} \rangle \vdash \langle o_i^n, o_i^{k+r} \rangle$; hence $E \vdash \langle o_i^n, o_j^{l+r} \rangle$. By the construction of T we get $k \leq n$.

If $l \geq m+d$, then $E \vdash \langle o_j^{l-d}, o_j^l \rangle$, so that $E \vdash \langle o_i^k, o_j^{l-d} \rangle$, which is a contradiction. We get $l < m+d$.

Firstly, let $k < n$. We shall prove that $T^{i,j}$ satisfies (3). If $l \geq m$, then

$$E \vdash \langle o_i^k, o_j^l \rangle, \langle o_j^l, o_j^{l+d} \rangle, \langle o_j^{l+d}, o_i^{k+d} \rangle,$$

and so $E \vdash \langle o_i^k, o_i^{k+d} \rangle$, which is a contradiction with $k < n$. We get $l < m$; it is sufficient to prove $n-k = m-l$. Suppose $n-k < m-l$.

Then $l + n - k < m$; as

$$E \vdash \langle o_j^{l+n-k}, o_i^n \rangle, \langle o_i^n, o_i^{n+d} \rangle, \langle o_i^{n+d}, o_j^{l+n-k+d} \rangle,$$

we have $E \vdash \langle o_j^{l+n-k}, o_j^{l+n-k+d} \rangle$, so that $m \leq l + n - k$, which is a contradiction. Suppose $m - l < n - k$. Then

$$E \vdash \langle o_i^{k+m-l}, o_j^m \rangle, \langle o_j^m, o_j^{m+d} \rangle, \langle o_j^{m+d}, o_i^{k+m-l+d} \rangle,$$

so that $n \leq k + m - l$, which is a contradiction.

Let, secondly, $k = n$. We shall prove that $T^{i,j}$ satisfies (4); it is sufficient to prove $l \geq m$. We have

$$E \vdash \langle o_j^l, o_i^n \rangle, \langle o_i^n, o_i^{n+d} \rangle, \langle o_i^{n+d}, o_j^{l+d} \rangle;$$

from this $l \geq m$ follows easily.

We have proved that T satisfies (5). Let us prove (6). If $T^{i,j}$ satisfies (1), then $T^{j,i}$ is evidently empty. Let $T^{i,j}$ satisfy (2). Then $T^{j,i} = \{\langle o_j^{\bar{l}}, o_i^{\bar{k}} \rangle\}$, where \bar{l} is the least number such that there exists a k_0 with $E \vdash \langle o_j^{\bar{l}}, o_i^{k_0} \rangle$, and \bar{k} is the least number such that $E \vdash \langle o_j^{\bar{l}}, o_i^{\bar{k}} \rangle$. We evidently have $\bar{l} \leq l$ and $k \leq \bar{k}$. Suppose $\bar{l} < l$. Then

$$E \vdash \langle o_i^k, o_j^{\bar{l}} \rangle, \langle o_j^{\bar{l}}, o_i^{\bar{k}+l-\bar{l}} \rangle,$$

so that $E \vdash \langle o_i^k, o_i^{\bar{k}+l-\bar{l}} \rangle$; as $\bar{k} + l - \bar{l} > k$, we get a contradiction with the emptiness of $T^{i,i}$. Hence $\bar{l} = l$; the equality $\bar{k} = k$ is now evident.

If $T^{i,j}$ satisfies (3), then the proof is similar to the previous one; let it satisfy (4). We have $T^{j,i} = \{\langle o_j^{\bar{l}}, o_i^{\bar{k}} \rangle\}$, where the numbers \bar{l} and \bar{k} are constructed as above. It is easy to prove that $E \vdash \langle o_j^m, o_i^{n+d-c} \rangle$. Evidently $\bar{l} \leq m$, $\bar{k} \geq n$. As

$$E \vdash \langle o_j^{\bar{l}}, o_i^{\bar{k}} \rangle, \langle o_i^{\bar{k}}, o_i^{\bar{k}+d} \rangle, \langle o_i^{\bar{k}+d}, o_j^{\bar{l}+d} \rangle,$$

we have $E \vdash \langle o_j^{\bar{l}}, o_j^{\bar{l}+d} \rangle$ and, consequently, $\bar{l} \geq m$, so that $\bar{l} = m$. We have $E \vdash \langle o_i^{\bar{k}}, o_i^{n+d-c} \rangle$; if we had $\bar{k} < n + d - c$, we should easily get a contradiction by Lemma 4; hence $\bar{k} = n + d - c$.

Let us prove that T satisfies (7). We evidently have $T^{i,h} = \{\langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle\}$, where \bar{k} is the least number such that there exists an l_0 with $E \vdash \langle o_i^{\bar{k}}, o_h^{l_0} \rangle$, and \bar{l} is the least number such that $E \vdash \langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle$.

Let $l < p$. As $E \vdash \langle o_i^{k+p-l}, o_h^q \rangle$, we get $\bar{k} \leq k + p - l$. Suppose $\bar{k} < k + p - l$. Put $\bar{\bar{k}} = \max(k, \bar{k})$, so that $\bar{\bar{k}} < k + p - l$. As

$$E \vdash \langle o_j^{l+\bar{\bar{k}}-k}, o_i^{\bar{\bar{k}}} \rangle, \langle o_i^{\bar{\bar{k}}}, o_h^{\bar{l}+\bar{\bar{k}}-\bar{k}} \rangle,$$

we get $p \leq l + \bar{\bar{k}} - k$, which is a contradiction. This proves $\bar{k} = k + p - l$. As $E \vdash \langle o_j^p, o_i^{\bar{k}} \rangle, \langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle$, we get $\bar{l} \geq q$ and from this $\bar{l} = q$ easily follows.

Let $l = p$. As $E \vdash \langle o_i^k, o_h^q \rangle$, we get $\bar{k} \leq k$. We have to prove $k - \bar{k} = q - \bar{l}$. Suppose $k - \bar{k} < q - \bar{l}$. As

$$E \vdash \langle o_i^p, o_i^k \rangle, \langle o_i^k, o_h^{\bar{l}+k-\bar{k}} \rangle,$$

we get $q \leq \bar{l} + k - \bar{k}$, which is a contradiction. Suppose $q - \bar{l} < k - \bar{k}$. Let $\bar{k} < k$ (if $\bar{k} = k$, we get a contradiction immediately). If $\bar{l} < q$, then

$$E \vdash \langle o_i^{\bar{k}+q-\bar{l}}, o_h^q \rangle, \langle o_h^q, o_j^p \rangle,$$

so that $k \leq \bar{k} + q - \bar{l}$, which is a contradiction. If we had $\bar{l} \geq q$, then we would have

$$E \vdash \langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle, \langle o_h^{\bar{l}}, o_j^{p+\bar{l}-q} \rangle,$$

so that $k \leq \bar{k}$, which is again a contradiction.

Let $p < l$. Then evidently $\bar{k} \leq k$.

Suppose first that the theories $T^{i,i}$, $T^{j,j}$ and $T^{h,h}$ are all empty. Suppose also $\bar{k} < k$. Then

$$E \vdash \langle o_h^{q+l-p}, o_i^k \rangle, \langle o_i^k, o_h^{\bar{l}+k-\bar{k}} \rangle,$$

so that $q + l - p = \bar{l} + k - \bar{k}$. If $q < \bar{l}$, then

$$E \vdash \langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle, \langle o_h^{\bar{l}}, o_j^{p+\bar{l}-q} \rangle,$$

so that $k \leq \bar{k}$, which is a contradiction. If we had $\bar{l} \leq q$, then we would have

$$E \vdash \langle o_i^{\bar{k}+q-\bar{l}}, o_h^q \rangle, \langle o_h^q, o_j^p \rangle,$$

where $\bar{k} + q - \bar{l} = k + p - l < k$, which is not possible. We thus get $\bar{k} = k$. As $E \vdash \langle o_i^k, o_h^{q+l-p} \rangle$, we get $\bar{l} = q + l - p$.

It remains to consider the case where $T^{i,i}$, $T^{j,j}$ and $T^{h,h}$ are all non-empty (if one is non-empty, then all are non-empty). There exist numbers n_1, n_2, n and d ($d > 0$) such that

$$T^{i,i} = \{ \langle o_i^{n_1}, o_i^{n_1+d} \rangle \},$$

$$T^{j,j} = \{ \langle o_j^{n_2}, o_j^{n_2+d} \rangle \},$$

$$T^{h,h} = \{ \langle o_h^n, o_h^{n+d} \rangle \}.$$

Let $q + l - p < n + d$. Suppose $\bar{k} < k$. We have $\bar{l} < q$, as in the opposite case we would have

$$E \vdash \langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle, \langle o_h^{\bar{l}}, o_j^{p+\bar{l}-q} \rangle,$$

which is a contradiction with $\bar{k} < k$. If we had $\bar{l} + k - \bar{k} \geq q + l - p$, then we would have $E \vdash \langle o_i^{\bar{k}+q-\bar{l}}, o_h^q \rangle, \langle o_h^q, o_j^p \rangle$ and $\bar{k} + q - \bar{l} < k$ (as

$\bar{k} + q - \bar{l} \leq k + p - l < k$), which is a contradiction. If we had $\bar{l} + k - \bar{k} < q + l - p$, then we would have

$$E \vdash \langle o_h^{\bar{l}+k-\bar{k}}, o_i^k \rangle, \langle o_i^k, o_h^{q+l-p} \rangle$$

and

$$\bar{l} + k - \bar{k} < q + l - p < n + d,$$

which is evidently a contradiction. Hence $\bar{k} = k$. We have $\bar{l} \leq q + l - p$. As

$$E \vdash \langle o_h^{\bar{l}}, o_i^{\bar{k}} \rangle, \langle o_i^{\bar{k}}, o_h^{q+l-p} \rangle$$

and $q + l - p < n + d$, we get $\bar{l} = q + l - p$.

Let $q + l - p > n + d$. Evidently $k = n_1, p = n_2, n < q + l - p - d < n + d$ and $E \vdash \langle o_i^k, o_h^{q+l-p-d} \rangle$. Suppose $\bar{k} < k$. We have $\bar{l} < n$, as in the opposite case we would have

$$E \vdash \langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle, \langle o_h^{\bar{l}}, o_j^{n_2+d-(q-n)+\bar{l}-n} \rangle,$$

so that $\bar{k} \geq k$. If $\bar{l} + k - \bar{k} \geq q + l - p - d$, then

$$E \vdash \langle o_i^{\bar{k}+n-\bar{l}}, o_h^n \rangle, \langle o_h^n, o_j^{n_2+d-(q-n)} \rangle$$

and $\bar{k} + n - \bar{l} < k$ (as $\bar{k} + n - \bar{l} < \bar{k} + q + l - p - d - \bar{l} \leq k$), which is a contradiction. If $\bar{l} + k - \bar{k} < q + l - p - d$, then

$$E \vdash \langle o_h^{q+l-p-d}, o_i^k \rangle, \langle o_i^k, o_h^{\bar{l}+k-\bar{k}} \rangle$$

and

$$\bar{l} + k - \bar{k} < q + l - p - d < n + d,$$

which is again a contradiction. We get $\bar{k} = k$. Evidently, $\bar{l} \leq q + l - p - d$. As $E \vdash \langle o_h^{\bar{l}}, o_i^{\bar{k}} \rangle, \langle o_i^{\bar{k}}, o_h^{q+l-p-d} \rangle$ and $q + l - p - d < n + d$, we get $\bar{l} = q + l - p - d$.

Let $q + l - p = n + d$. Evidently $k = n_1, p = n_2, E \vdash \langle o_i^k, o_h^n \rangle$. It is sufficient to prove that $k - \bar{k} = n - \bar{l}$. If we had $k - \bar{k} < n - \bar{l}$, then we would have

$$E \vdash \langle o_h^{\bar{l}+k-\bar{k}}, o_i^k \rangle, \langle o_i^k, o_h^n \rangle,$$

where $\bar{l} + k - \bar{k} < n$, which is again a contradiction. If $n - \bar{l} < k - \bar{k}$ and, simultaneously, $\bar{l} \leq n$, then

$$E \vdash \langle o_i^{\bar{k}+n-\bar{l}}, o_h^n \rangle, \langle o_h^n, o_j^{n_2+d-(q-n)} \rangle,$$

so that $\bar{k} + n - \bar{l} \geq k$, which is a contradiction. If we had $\bar{l} > n$, then we would have

$$E \vdash \langle o_i^{\bar{k}}, o_h^{\bar{l}} \rangle, \langle o_h^{\bar{l}}, o_j^{n_2+d-(q-n)+\bar{l}-n} \rangle,$$

so that $\bar{k} \geq k$ and, consequently, $\bar{k} = k$; but then it is easy to see that $\bar{l} \leq n$; we get again a contradiction. We have thus proved $k - \bar{k} = n - \bar{l}$.

LEMMA 6. *Each constant theory E is equivalent to a constant-reduced theory; namely, it is equivalent to $\mathcal{R}_c(E)$.*

Proof. Put $T = \mathcal{R}_c(E)$. It is sufficient to prove $T \vdash E$. Suppose on the contrary that there exists an equation $a \in E$ such that $T \not\vdash a$ does not hold.

Let, firstly, $a = \langle o_i^r, o_i^s \rangle$, where $i \in I$. Evidently $r \neq s$; we have $T^{i,i} = \{\langle o_i^n, o_i^{n+d} \rangle\}$, where $d > 0$. By Lemma 4 there exists an equation $\langle o_i^p, o_i^q \rangle$ such that $\{a, \langle o_i^n, o_i^{n+d} \rangle\} \vdash \langle o_i^p, o_i^q \rangle$ and either $p = n \neq q$ and $q < n + d$ or $p < n$ and $p \neq q$. As $E \vdash T$ and $a \in E$, we have $E \vdash \langle o_i^p, o_i^q \rangle$; this is a contradiction with the construction of $\mathcal{R}_c(E)$.

Secondly, let $a = \langle o_i^r, o_j^s \rangle$, where $i, j \in I$ and $i \neq j$. There are three possible cases:

(a) $T^{i,j} = \{\langle o_i^k, o_j^l \rangle\}$. As $E \vdash \langle o_i^r, o_j^s \rangle$, we have $r \geq k$. As $E \vdash T^{i,j} \vdash \langle o_j^{l+r-k}, o_i^r \rangle$ and $E \vdash \langle o_i^r, o_j^s \rangle$, we get $E \vdash \langle o_j^{l+r-k}, o_j^s \rangle$; by the construction of $\mathcal{R}_c(E)$ we have $l + r - k = s$. From this it follows that $T \vdash a$, which is a contradiction.

(b) $T^{i,j} = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle, \langle o_i^{n-c}, o_j^{m-c} \rangle\}$, where $d, c > 0$. As $E \vdash \langle o_i^r, o_j^s \rangle$, we have $r \geq n - c$. We also have $r - (n - c) \neq s - (m - c)$, as in the opposite case we would have $T \vdash a$. Since $E \vdash \langle o_i^r, o_i^{n-c+s-(m-c)} \rangle$, we get $r \geq n$. Evidently $E \vdash \langle o_j^s, o_j^{m+r-n} \rangle$, so that $s \geq m$. Let c_1 (c_2 , resp.) be the least non-negative integer such that d divides $r - (n + c_1)$ (d divides $s - (m + c_2)$, resp.). Evidently $c_1, c_2 < d$. Since

$$E \vdash \langle o_j^{m+c_1}, o_i^{n+c_1} \rangle, \langle o_i^{n+c_1}, o_i^r \rangle, \langle o_i^r, o_j^s \rangle, \langle o_j^s, o_j^{m+c_2} \rangle,$$

we get $E \vdash \langle o_j^m, o_j^{m+d-|c_1-c_2|} \rangle$, where $0 < d - |c_1 - c_2| \leq d$; by the construction of $\mathcal{R}_c(E)$ we get $d - |c_1 - c_2| = d$, so that $c_1 = c_2$. From this it follows that $T \vdash a$, which is a contradiction.

(c) $T^{i,j} = \{\langle o_i^n, o_i^{n+d} \rangle, \langle o_j^m, o_j^{m+d} \rangle, \langle o_i^n, o_j^{m+c} \rangle\}$, where $d > 0$ and $0 \leq c < d$. As $E \vdash \langle o_i^r, o_j^s \rangle$, we have $r \geq n$. Since $E \vdash \langle o_j^s, o_j^{m+c+r-n} \rangle$, we have $s \geq m$. Let c_1 (c_2 , resp.) be the least non-negative integer such that d divides $r - (n + c_1)$ (d divides $s - (m + c + c_1)$, resp.). Since

$$E \vdash \langle o_j^{m+c+c_1}, o_i^{n+c_1} \rangle, \langle o_i^{n+c_1}, o_i^r \rangle, \langle o_i^r, o_j^s \rangle, \langle o_j^s, o_j^{m+c+c_2} \rangle,$$

we get $E \vdash \langle o_j^{m+c_1}, o_j^{m+c_2} \rangle$; now a contradiction can be obtained as in the previous case.

LEMMA 7. *Let $E = \{\langle x^n, x^{n+d} \rangle\}$, where $d > 0$. Let M be the set of all $\langle z^m, z^{m+e} \rangle$, where $z \in Y$, $n \leq m$, $e > 0$ and d divides e . Then $\text{Cn } E = \text{Triv} \cup M \cup M^{-1}$.*

Proof. It is easy to prove that $\text{Triv} \cup M \cup M^{-1}$ is a fully invariant congruence relation of W ; our assertion is then evident.

LEMMA 8. *Let $E = \{\langle x^n, y^n \rangle\}$. Let M be the set of all $\langle z^k, w^l \rangle$, where $z \in Y$, $w \in Y$, $k \geq n$ and $l \geq n$. Then $\text{Cn } E = \text{Triv} \cup M$.*

The proof is analogous to the previous one.

LEMMA 9. *Each variable theory E is equivalent to a variable-reduced theory.*

Proof. Similarly as Lemma 4, the following statement can be proved: If $\alpha = \langle x^n, x^{n+d} \rangle$ and $\beta = \langle x^m, x^{m+e} \rangle$, where $d, e > 0$, then either $\alpha \vdash \beta$ or there exists an equation $\gamma = \langle x^p, x^q \rangle$ such that $\{\alpha, \beta\} \vdash \gamma$ and either $p = n \neq q$ and $q < n + d$ or $p < n$ and $p \neq q$. From this the following follows easily:

- (25) *Each theory which contains only the equations $\langle x^k, x^l \rangle$ with various k and l is equivalent either to the empty theory or to a single equation $\langle x^n, x^{n+d} \rangle$, where $d > 0$.*

Let us prove that

- (26) *An equation $\langle x^n, y^{n+d} \rangle$, where $d > 0$, is equivalent to $\langle x^n, y^n \rangle$. Similarly, $\langle x^{n+d}, y^n \rangle$ is equivalent to $\langle x^n, y^n \rangle$.*

In fact, we have $\langle x^n, y^{n+d} \rangle \vdash \langle y^n, y^{n+d} \rangle$ and, consequently, $\langle x^n, y^{n+d} \rangle \vdash \langle y^{n+d}, y^n \rangle$; by transitivity we have $\langle x^n, y^{n+d} \rangle \vdash \langle x^n, y^n \rangle$. Conversely, $\langle x^n, y^n \rangle \vdash \langle x^n, y^{n+d} \rangle$ is evident. Thus (26) is proved.

Let us prove that

- (27) *Each theory which contains only the equations $\langle x^k, y^l \rangle$ with various k and l is either empty or equivalent to a single equation $\langle x^n, y^n \rangle$.*

In fact, by (26) we may suppose that the theory contains only the equations $\langle x^k, y^k \rangle$ (with various k); then n may be chosen as the least number such that $\langle x^n, y^n \rangle$ belongs to that theory.

Each variable theory E is evidently equivalent to a theory containing only the equations $\langle x^k, x^l \rangle$ and $\langle x^k, y^l \rangle$. To prove Lemma 9 it is thus sufficient (by (25) and (27)) to prove that if $\alpha = \langle x^n, x^{n+d} \rangle$, where $d > 0$ and $\beta = \langle x^p, y^p \rangle$, then $\{\alpha, \beta\}$ is equivalent to a variable-reduced theory. If $p \leq n$, then $\beta \vdash \alpha$, so that $\{\alpha, \beta\}$ is equivalent to $\{\beta\}$. Let $p > n$. We shall prove that $\{\alpha, \beta\}$ is equivalent to $\{\langle x^n, y^n \rangle\}$. We evidently have $\langle x^n, y^n \rangle \vdash \{\alpha, \beta\}$; it is now sufficient to prove that $\{\alpha, \beta\} \vdash \langle x^n, y^n \rangle$. Let c be the least non-negative integer such that $p - cd < n + d$; put $r = p - cd$. We have $n \leq r < n + d$ and $\alpha \vdash \langle x^p, x^r \rangle$, $\beta \vdash \langle y^p, y^r \rangle$. Hence $\{\alpha, \beta\} \vdash \langle x^r, y^r \rangle$. As $r < n + d$, we get $\{\alpha, \beta\} \vdash \langle x^{n+d}, y^{n+d} \rangle$ and, consequently, $\{\alpha, \beta\} \vdash \langle x^n, y^n \rangle$.

LEMMA 10. *Two variable-reduced theories are equivalent if and only if they are equal.*

Lemma 10 easily follows from Lemmas 7 and 8.

From Lemmas 9 and 10 it follows that to each theory E there exists exactly one variable-reduced theory T which is equivalent to $V \cap \text{Cn} E$; the theory T will be denoted by $\mathcal{R}_v(E)$.

LEMMA 11. Let E be a reduced theory such that $E \cap V = \{\langle x^n, x^{n+d} \rangle\}$, where $d > 0$. Then $\text{Cn} E = \text{Cn}(E \cap V) \cup \text{Cn}(E \cap C)$.

Proof. It is sufficient to prove that $\text{Cn}(E \cap V) \cup \text{Cn}(E \cap C)$ is a fully invariant congruence relation of W ; to prove this it is sufficient to prove that if $\langle a, b \rangle \in \text{Cn}(E \cap V)$, $\langle b, c \rangle \in \text{Cn}(E \cap C)$ and $b \neq c$, then $\langle a, c \rangle \in \text{Cn}(E \cap C)$. By Lemmas 2 and 3 we have $b = o_i^k$ and $c = o_j^l$, where $i, j \in I$. By Lemma 7 we get $a = o_i^p$ and $n \leq p$, $n \leq k$ and d divides $p - k$. We have $E^{i,i} = \{\langle o_i^m, o_i^{m+e} \rangle\}$, where $m \leq n$ and e divides d . Hence $m \leq p$, $m \leq k$ and e divides $p - k$, so that $\langle a, b \rangle \in \text{Cn} E^{i,i} \subseteq \text{Cn}(E \cap C)$. As $\langle b, c \rangle \in \text{Cn}(E \cap C)$, we get $\langle a, c \rangle \in \text{Cn}(E \cap C)$.

LEMMA 12. Let E be a reduced theory such that $E \cap V = \{\langle x^n, y^n \rangle\}$. Let us define m_i for each $i \in I$ by $E^{i,i} = \{\langle o_i^{m_i}, o_i^{m_i+1} \rangle\}$. Let M be the set of all $\langle z^k, o_i^l \rangle$ such that $z \in X$, $i \in I$, $k \geq n$ and $l \geq m_i$. Then

$$\text{Cn} E = \text{Cn}(E \cap V) \cup \text{Cn}(E \cap C) \cup M \cup M^{-1}.$$

Proof. Put $T = \text{Cn}(E \cap V) \cup \text{Cn}(E \cap C) \cup M \cup M^{-1}$. It is sufficient to prove that T is a fully invariant congruence relation of W . T is evidently a reflexive and symmetrical relation. Let us prove the transitivity.

Let $\langle a, b \rangle \in T$ and $\langle b, c \rangle \in T$; we have to prove $\langle a, c \rangle \in T$. Consider the following cases (the other cases are either trivial or analogous to the considered ones):

(a) $\langle a, b \rangle \in \text{Cn}(E \cap V)$ and $\langle b, c \rangle \in \text{Cn}(E \cap C)$. Then $b = o_i^k$ and $c = o_j^l$, where $i, j \in I$. We have $a = z^p$, where $z \in Y$. By Lemma 8 we get $n \leq p$ and $n \leq k$. Hence $m_i \leq k$; from Lemmas 2 and 3 it easily follows that $m_j \leq l$. Consequently, if $z \in X$, then $\langle a, c \rangle \in M$. Let $z = o_h$, where $h \in I$. If $h = i$, then we have $\langle a, b \rangle \in \text{Cn} E^{i,i} \subseteq \text{Cn}(E \cap C)$ and, consequently, $\langle a, c \rangle \in \text{Cn}(E \cap C)$. Let $h \neq i$. We have $E^{h,i} = \{\langle o_h^r, o_i^s \rangle\}$. As $E^{h,h} = \{\langle o_h^{m_h}, o_h^{m_h+1} \rangle\}$ and $E^{i,i} = \{\langle o_i^{m_i}, o_i^{m_i+1} \rangle\}$, we have $0 \leq m_h - r = m_i - s$ and, consequently, $\langle o_h^{m_h}, o_i^{m_i} \rangle \in \text{Cn} E^{h,i} \subseteq \text{Cn}(E \cap C)$; from this $\langle a, c \rangle \in \text{Cn}(E \cap C)$ follows.

(b) $\langle a, b \rangle \in \text{Cn}(E \cap V)$ and $\langle b, c \rangle \in M$. Then $b = z^k$ and $c = o_i^l$, where $z \in X$, $k \geq n$ and $l \geq m_i$. By Lemma 8 we have $a = w^p$, where $w \in Y$ and $p \geq n$. If $w \in X$, then $\langle a, c \rangle \in M$; if $w \in O$, then $\langle a, c \rangle \in \text{Cn}(E \cap C)$ can be proved similarly as in the previous case.

(c) $\langle a, b \rangle \in \text{Cn}(E \cap V)$ and $\langle b, c \rangle \in M^{-1}$. Then $b = o_i^l$ and $c = z^k$, where $z \in X$, $k \geq n$ and $l \geq m_i$. By Lemma 8 we have $a = w^p$, where $w \in Y$ and $p \geq n$. Evidently, $\langle a, c \rangle \in \text{Cn}(E \cap V)$.

(d) $\langle a, b \rangle \in \text{Cn}(E \cap C)$ and $\langle b, c \rangle \in M^{-1}$. Then $b = o_i^l$ and $c = z^k$, where $z \in X$, $k \geq n$ and $l \geq m_i$. By Lemmas 2 and 3 we have $a = o_j^p$, where $j \in I$ and $p \geq m_j$. We get $\langle a, c \rangle \in M^{-1}$.

(e) $\langle a, b \rangle \in M$ and $\langle b, c \rangle \in M^{-1}$. Then $\langle a, c \rangle \in \text{Cn}(E \cap V)$.

(f) $\langle a, b \rangle \in M^{-1}$ and $\langle b, c \rangle \in M$. Then $a = o_i^k$ and $c = o_j^l$, where $k \geq m_i$ and $l \geq m_j$. We can prove $\langle a, c \rangle \in \text{Cn}(E \cap C)$ similarly as in the case (a).

We have proved the transitivity. T is evidently a congruence relation. Let φ be an endomorphism of W and $\langle a, b \rangle \in T$; we have to prove $\langle \varphi(a), \varphi(b) \rangle \in T$. If $\langle a, b \rangle \in \text{Cn}(E \cap V) \cup \text{Cn}(E \cap C)$, then it is evident. Let e.g. $\langle a, b \rangle \in M$. Then $a = z^k$ and $b = o_i^l$, where $z \in X$, $k \geq n$ and $l \geq m_i$. There exists a $w \in Y$ such that $\varphi(z) = w^p$. We have $\langle \varphi(a), \varphi(b) \rangle = \langle w^{k+p}, o_i^l \rangle$. If $w \in X$, then $\langle \varphi(a), \varphi(b) \rangle \in M$. If $w \in O$, then $\langle \varphi(a), \varphi(b) \rangle \in \text{Cn}(E \cap C)$.

If E is an arbitrary theory, then put $\mathcal{R}(E) = \mathcal{R}_c(E) \cup \mathcal{R}_v(E)$.

LEMMA 13. $\mathcal{R}(E)$ is a reduced theory.

Proof. Put $T = \mathcal{R}(E)$. Conditions (11), (12) and (13) are evidently satisfied.

Let $T \cap V = \{\langle x^n, x^{n+d} \rangle\}$. Let $i \in I$. We have

$$T^{i,i} = (\mathcal{R}_c(E))^{i,i} = \{\langle o_i^m, o_i^{m+e} \rangle\}.$$

As $E \vdash \langle o_i^n, o_i^{n+d} \rangle$, it easily follows from Lemma 4 and from the construction of $\mathcal{R}_c(E)$ that $\langle o_i^m, o_i^{m+e} \rangle \vdash \langle o_i^n, o_i^{n+d} \rangle$; by Lemma 1 we get $m \leq n$ and e divides d .

Let $T \cap V = \{\langle x^n, y^n \rangle\}$. If $i, j \in I$ and $i \neq j$, then $E \vdash \langle o_i^n, o_j^n \rangle$, so that $T^{i,j}$ is non-empty. Let $i \in I$. We have $E \vdash \langle o_i^n, o_i^{n+1} \rangle$. Hence $T^{i,i} = \{\langle o_i^m, o_i^{m+e} \rangle\}$. From Lemma 4 and from the construction of $\mathcal{R}_c(E)$ it easily follows that $\langle o_i^m, o_i^{m+e} \rangle \vdash \langle o_i^n, o_i^{n+1} \rangle$. By Lemma 1 we get $m \leq n$ and e divides 1; this implies $e = 1$.

LEMMA 14. Each theory E is equivalent to a reduced theory; namely, it is equivalent to $\mathcal{R}(E)$.

Proof. Let us prove first the following statement:

(28) If E is a theory and α a constant equation, then $E \vdash \alpha$ if and only if $C \cap \text{Cn} E \vdash \alpha$.

If $E \vdash \alpha$, then $\alpha \in \text{Cn} E$, so that $\alpha \in C \cap \text{Cn} E$ and $C \cap \text{Cn} E \vdash \alpha$. If $C \cap \text{Cn} E \vdash \alpha$, then $\text{Cn} E \vdash \alpha$ and, consequently, $E \vdash \alpha$. (28) is thus proved.

Let E be a theory. E is equivalent to $\text{Cn} E$; as each equation $\langle z^n, o_i^m \rangle$, where $z \in X$, is evidently equivalent to $\{\langle x^n, y^n \rangle, \langle o_i^n, o_i^m \rangle\}$, the theory $\text{Cn} E$ is equivalent to $(V \cap \text{Cn} E) \cup (C \cap \text{Cn} E)$. The theory $V \cap \text{Cn} E$ is equivalent to $\mathcal{R}_v(E)$. On the other hand, $C \cap \text{Cn} E$ is equivalent by Lemma 6 to $\mathcal{R}_c(C \cap \text{Cn} E)$; by the definition of \mathcal{R}_c and by (28) we get $\mathcal{R}_c(C \cap \text{Cn} E) = \mathcal{R}_c(E)$. Hence, E is equivalent to $\mathcal{R}_c(E) \cup \mathcal{R}_v(E) = \mathcal{R}(E)$.

LEMMA 15. Let E_1 and E_2 be two reduced theories. Then $E_1 \vdash E_2$ if and only if $E_1 \cap V \vdash E_2 \cap V$ and $E_1 \cap C \vdash E_2 \cap C$.

Proof. Let $E_1 \vdash E_2$. By Lemmas 2 and 3 we have $\text{Cn}(E_1 \cap C) \subseteq C \cup \text{Triv}$; hence, by Lemmas 11 and 12, we have $E_1 \cap V \vdash E_2 \cap V$. By the definition of a reduced theory and by Lemmas 7 and 8 we have $C \cap \text{Cn}(E_1 \cap V) \subseteq \text{Cn}(E_1 \cap C)$. From this and from Lemmas 11 and 12 we get $E_1 \cap C \vdash E_2 \cap C$. The converse implication is evident.

LEMMA 16. *Two reduced theories are equivalent if and only if they are equal.*

Proof. From Lemmas 2 and 3 it easily follows that two constant-reduced theories are equivalent if and only if they are equal. The rest follows from this and from Lemmas 10 and 15.

Theorem 2 easily follows from Lemmas 14 and 16.

Remark. Let I be a finite set. Applying Lemmas 1, 2, 3, 7, 8, 11 and 12 we may construct an algorithm which for each two reduced theories E_1 and E_2 decides whether $E_1 \vdash E_2$ holds or does not hold. (We have $E_1 \vdash E_2$ if and only if $\alpha \in \text{Cn} E_1$ for each $\alpha \in E_2$.) Hence the lattice L in Theorem 2 is defined constructively and we may assert that a complete description of the lattice $\mathcal{L}_{(I,1)}$ is given.

4. Lattices $\mathcal{L}_{((0,1))}$ and $\mathcal{L}_{((1,1))}$. Let us apply Theorem 2 to the case $\text{Card } I \leq 1$ and let us give a more simple description of the lattices $\mathcal{L}_{((0,1))}$ and $\mathcal{L}_{((1,1))}$.

Let D be the partially ordered set of all natural numbers with the partial ordering r defined by $\langle n, m \rangle \in r$ if and only if either $n = 0$ or $n \neq 0$, $m \neq 0$ and n divides m . Evidently, D is a distributive lattice. Let ω be the set of all natural numbers with the usual ordering of natural numbers; ω is a distributive lattice. The direct product $\omega \times D$ is again a distributive lattice; let us denote it by K . The lattice K has no greatest element; if we add a new element ∞_0 to it and declare ∞_0 to be the greatest element, then the resulting set is again a distributive lattice; we denote it by L_0 .

Let us denote by M the direct product $L_0 \times K$. Let N_1 be the set of all the elements $\langle \langle n, d \rangle, \langle m, e \rangle \rangle$ of M such that $m \leq n$, $e > 0$, e divides d and $d = 0$ implies $e = 1$. Let N_2 be the set of all the elements $\langle \infty_0, \langle m, e \rangle \rangle$ of M such that $e > 0$. It is easy to see that the set $N = N_1 \cup N_2$ is a sublattice of M ; hence N is a distributive lattice. It has no greatest element; if we add a new element ∞_1 to it and declare ∞_1 to be the greatest element, then the resulting set is again a distributive lattice; let us denote it by L_1 .

By Theorem 2 the following two theorems can be proved:

THEOREM 3. *The lattice $\mathcal{L}_{((0,1))}$ is isomorphic to L_0 . The isomorphism can be established in the following way:*

(a) ∞_0 corresponds to the class of all algebras of type $((0, 1))$;

(b) an element $\langle n, d \rangle$ with $d > 0$ corresponds to the class of all models of $\langle x^n, x^{n+d} \rangle$;

(c) an element $\langle n, 0 \rangle$ corresponds to the class of all models of $\langle x^n, y^n \rangle$.

THEOREM 4. The lattice $\mathcal{L}_{((1,1))}$ is isomorphic to L_1 . The isomorphism can be established in the following way:

(a) ∞_1 corresponds to the class of all algebras of type $((1, 1))$;

(b) an element $\langle \infty_0, \langle m, e \rangle \rangle$ corresponds to the class of all models of $\langle o^m, o^{m+e} \rangle$ (where o is the nullary operation);

(c) an element $\langle \langle n, d \rangle, \langle m, e \rangle \rangle$ with $d > 0$ corresponds to the class of all models of $\{\langle x^n, x^{n+d} \rangle, \langle o^m, o^{m+e} \rangle\}$;

(d) an element $\langle \langle n, 0 \rangle, \langle m, 1 \rangle \rangle$ corresponds to the class of all models of $\{\langle x^n, y^n \rangle, \langle o^m, o^{m+e} \rangle\}$.

COROLLARY. The lattices $\mathcal{L}_{((0,1))}$ and $\mathcal{L}_{((1,1))}$ are distributive.

5. Cardinality of \mathcal{L}_A . Consider algebras with two unary operations; we shall denote these operations by x' and x^+ . If a is an element of such an algebra and $s = \langle s_1, s_2, \dots, s_n \rangle$ (where $n \geq 0$) is a finite sequence of symbols $|$ and $+$, then a^s is defined inductively: if s is empty, then $a^s = a$; if $a^{\langle s_1, \dots, s_n \rangle}$ is already defined, then $a^{\langle s_1, \dots, s_n, | \rangle} = (a^{\langle s_1, \dots, s_n \rangle})'$ and $a^{\langle s_1, \dots, s_n, + \rangle} = (a^{\langle s_1, \dots, s_n \rangle})^+$.

THEOREM 5. The lattice $\mathcal{L}_{((0,2))}$ has exactly 2^{\aleph_0} elements. Moreover, there are 2^{\aleph_0} primitive classes of algebras with two unary operations which satisfy the equation

$$(1) \quad x'' = y''.$$

Proof. Let M be the set of all infinite sequences $e = \langle e_1, e_2, \dots \rangle$ of numbers 0 and 1, so that M has 2^{\aleph_0} elements.

Let A be an infinite countable set; let us denote its elements by $a_1, b_1, a_2, b_2, a_3, b_3, \dots$. We shall define an algebra A_e for each $e \in M$ in this way: its elements are exactly the elements of A , and the operations are defined by

$$(2) \quad \text{if } e_n = 0, \text{ then } a_n^+ = a_{n+1} \text{ and } a_n' = b_n' = b_n^+ = a_1 \text{ in } A_e;$$

$$(3) \quad \text{if } e_n = 1, \text{ then } a_n' = a_1, a_n^+ = b_n, b_n' = a_{n+1} \text{ and } b_n^+ = a_1 \text{ in } A_e.$$

If $e \in M$ and $n \geq 1$, then we define a finite sequence $s(e, n)$ inductively in this way: $s(e, 1)$ is the empty sequence; if $s(e, n) = \langle s_1, \dots, s_k \rangle$, then $s(e, n+1) = \langle s_1, \dots, s_k, + \rangle$ in the case of $e_n = 0$ and $s(e, n+1) = \langle s_1, \dots, s_k, +, 1 \rangle$ in the case of $e_n = 1$. By induction on n we can prove that

$$(4) \quad \text{if } e \in M \text{ and } n \geq 1, \text{ then } a_1^{s(e, n)} = a_n \text{ in } A_e.$$

If $e \in M$, then let \mathfrak{U}_e be the primitive class generated by the algebra A_e , i.e. the class of all algebras satisfying all the equations which are satisfied in A_e .

By (2) and (3) we get

$$(5) \quad \text{if } a \in A, \text{ then } a'' = a_1 \text{ holds in } A_e.$$

Hence in A_e and, consequently, in \mathfrak{U}_e equation (1) is satisfied. As there are at most 2^{\aleph_0} primitive classes, it is now sufficient to prove that $\mathfrak{U}_e \neq \mathfrak{U}_{e^*}$ for all $e, e^* \in M$ such that $e \neq e^*$.

Let $e, e^* \in M$, $e \neq e^*$. Let n be the least number such that $e_n \neq e_n^*$. We may suppose

$$(6) \quad e_n = 0 \quad \text{and} \quad e_n^* = 1$$

(in the opposite case the proof would be analogous). Put $s = s(e, n)$, so that $s = s(e^*, n)$. By (4) we get

$$(7) \quad a_1^s = a_n \text{ holds both in } A_e \text{ and } A_{e^*}.$$

By (2), (3) and (6) we get

$$(8) \quad a_n^{+'} = a_1 \quad \text{in } A_e$$

and

$$(9) \quad a_n^{+'} = a_{n+1} \quad \text{in } A_{e^*}.$$

Consider the equation $\langle ((x'')^s)^{+'}, x'' \rangle$. By (5), (7) and (8) this equation is satisfied in A_e . By (5), (7) and (9) this equation is not satisfied in A_{e^*} . Hence $\mathfrak{U}_e \neq \mathfrak{U}_{e^*}$.

Kalicki [2] has proved that the cardinal number of the lattice $\mathcal{L}_{((0,0,1))}$ is 2^{\aleph_0} (and, moreover, that the lattice has 2^{\aleph_0} atoms). Combining this with our results, one can easily prove the following

THEOREM 6. *Let a type Δ be given. The lattice \mathcal{L}_Δ is finite if and only if $\Delta = ((k))$, where k is a natural number. The lattice \mathcal{L}_Δ is countably infinite if and only if $\Delta = ((k, 1))$, where k is a natural number.*

6. Distributivity and modularity of lattices \mathcal{L}_Δ . We now prove

THEOREM 7. *Let a type Δ be given. The lattice \mathcal{L}_Δ is distributive if and only if either $\Delta = ((k))$, where $k \leq 2$, or $\Delta = ((k, 1))$, where $k \leq 1$. The lattice \mathcal{L}_Δ is modular if and only if either $\Delta = ((k))$, where $k \leq 3$, or $\Delta = ((k, 1))$, where $k \leq 1$.*

Proof. The case $\Delta = ((I))$, where I is a set was discussed in Corollary 2 of Theorem 1. Let Δ contain not only nullary operations. If $\Delta = ((0, 1))$ or $\Delta = ((1, 1))$, then \mathcal{L}_Δ is distributive by the Corollary of Theorems 3 and 4. Let $\Delta \neq ((0, 1))$ and $\Delta \neq ((1, 1))$. We have to prove that \mathcal{L}_Δ is not modular. It is easy to see that \mathcal{L}_Δ contains a sublattice which is

isomorphic to one of the four lattices $\mathcal{L}_{((2,1))}$, $\mathcal{L}_{((0,2))}$, $\mathcal{L}_{((0,0,1))}$ and $\mathcal{L}_{((1,0,1))}$. Hence it is sufficient to prove the following four lemmas.

LEMMA 17. *The lattice $\mathcal{L}_{((2,1))}$ is not modular.*

Proof. Algebras of type $((2,1))$ have two nullary operations o_1 and o_2 and one unary operation x' . Let us define five primitive classes:

\mathfrak{U}_a is the class of all models of $\{\langle o_1, o_2 \rangle, \langle o_1, o'_1 \rangle\}$;

\mathfrak{U}_b is the class of all models of $\langle o_1, o_2 \rangle$;

\mathfrak{U}_c is the class of all models of $\{\langle o_1, o'_1 \rangle, \langle o_2, o'_2 \rangle\}$;

\mathfrak{U}_d is the class of all models of $\langle o_1, o'_1 \rangle$;

\mathfrak{U}_e is the class of all algebras of type $((2,1))$.

It is sufficient to prove that these primitive classes constitute a sublattice of $\mathcal{L}_{((2,1))}$ which is isomorphic to the lattice in Fig. 4. It follows from Lemmas 1 and 2 that each equation α such that $\langle o_1, o_2 \rangle \vdash \alpha$ and $\{\langle o_1, o'_1 \rangle, \langle o_2, o'_2 \rangle\} \vdash \alpha$ is trivial; hence the join of \mathfrak{U}_b and \mathfrak{U}_c in $\mathcal{L}_{((2,1))}$ is \mathfrak{U}_e . We evidently have $\mathfrak{U}_b \cap \mathfrak{U}_d = \mathfrak{U}_a$. By Lemma 1 we have $\mathfrak{U}_c \neq \mathfrak{U}_d$. Everything else is evident.

LEMMA 18. *The lattice $\mathcal{L}_{((0,2))}$ is not modular.*

Proof. Algebras of type $((0,2))$ have two unary operations x' and x^+ . Let us define five primitive classes:

\mathfrak{U}_a is the class of all models of $\{\langle x, x' \rangle, \langle x, x^+ \rangle\}$;

\mathfrak{U}_b is the class of all models of $\langle x, x' \rangle$;

\mathfrak{U}_c is the class of all models of $\langle x, x^+ \rangle$;

\mathfrak{U}_d is the class of all models of $\{\langle x'^+, x' \rangle, \langle x'^+, x^+ \rangle\}$;

\mathfrak{U}_e is the class of all models of $\langle x'^+, x^+ \rangle$.

It is sufficient to prove that these primitive classes constitute a sublattice of $\mathcal{L}_{((0,2))}$ which is isomorphic to the lattice in Fig. 4. The set E_1 of all equations which are valid in \mathfrak{U}_b is evidently just the set of all $\langle z^{s_1, \dots, s_n}, z^{t_1, \dots, t_m} \rangle$, where z is a variable and symbol $+$ has the same number of occurrences in s_1, \dots, s_n as in t_1, \dots, t_m . The set E_2 of all equations which are valid in \mathfrak{U}_c could be described similarly. Thus $E_1 \cap E_2$ is just the set of all $\langle z^{s_1, \dots, s_n}, z^{t_1, \dots, t_n} \rangle$, where z is a variable and the sequence s_1, \dots, s_n differs from t_1, \dots, t_n only by the order of its members; hence $E_1 \cap E_2$ is equivalent to $\langle x'^+, x^+ \rangle$. On the other hand, the class of all models of $E_1 \cap E_2$ is just the join of \mathfrak{U}_b and \mathfrak{U}_c in $\mathcal{L}_{((0,2))}$, so that this join is equal to \mathfrak{U}_e . We evidently have $\mathfrak{U}_b \cap \mathfrak{U}_d = \mathfrak{U}_a$. It is easy to prove that the set of all $\langle z^{s_1, \dots, s_n}, z^{t_1, \dots, t_m} \rangle$, where z is a variable and either $n = 0$ and $m = 0$ or $n \neq 0$ and $m \neq 0$ is a fully invariant congruence relation of $W_{((0,2))}$; from this we get $\mathfrak{U}_c \neq \mathfrak{U}_d$. Everything else is evident.

LEMMA 19. *The lattice $\mathcal{L}_{((0,0,1))}$ is not modular. Moreover, the lattice of all primitive classes of semigroups is not modular ⁽¹⁾.*

⁽¹⁾ I have not found this fact in literature.

Proof. Algebras of type $((0, 0, 1))$ have one binary operation xy . Let us define five primitive classes:

\mathfrak{A}_a is the class of all models of $\langle xy, zw \rangle$;

\mathfrak{A}_b is the class of all models of $\langle xy, xz \rangle$;

\mathfrak{A}_c is the class of all models of $\{\langle (xy)z, x(yz) \rangle, \langle xy, yx \rangle\}$;

\mathfrak{A}_d is the class of all models of $\{\langle (xy)z, x(yz) \rangle, \langle x(yz), x(zy) \rangle, \langle x(yz), (yz)x \rangle\}$;

\mathfrak{A}_e is the class of all models of $\{\langle (xy)z, x(yz) \rangle, \langle x(yz), x(zy) \rangle\}$.

It is sufficient to prove that these primitive classes constitute a sublattice of $\mathcal{L}_{((0,0,1))}$ which is isomorphic to the lattice in Fig. 4. If $w \in W_{((0,0,1))}$, then let us define a finite sequence $\|w\|$ of variables in the following way:

(a) if z is a variable, then $\|z\|$ is the sequence with a single member z ;

(b) if $\|w_1\| = (z_1, \dots, z_n)$ and $\|w_2\| = (t_1, \dots, t_m)$, then $\|w_1 w_2\| = (z_1, \dots, z_n, t_1, \dots, t_m)$.

The set E_1 of all equations which are valid in \mathfrak{A}_b is evidently just the set of all $\langle w_1, w_2 \rangle$ such that the first member of $\|w_1\|$ coincides with the first member of $\|w_2\|$. The set E_2 of all equations which are valid in \mathfrak{A}_c is evidently just the set of all $\langle w_1, w_2 \rangle$ such that the sequence $\|w_1\|$ differs from $\|w_2\|$ only by the ordering of its members. From this it can be shown similarly as in Lemma 18 that the join of \mathfrak{A}_b and \mathfrak{A}_c in $\mathcal{L}_{((0,0,1))}$ is equal to \mathfrak{A}_e . We evidently have $\mathfrak{A}_b \cap \mathfrak{A}_d = \mathfrak{A}_a$. It is easy to prove that the set of all $\langle w_1, w_2 \rangle$ such that either $w_1 = w_2$ or both $\|w_1\|$ and $\|w_2\|$ have at least three members is a fully invariant congruence relation of $W_{((0,0,1))}$; from this we get $\mathfrak{A}_c \neq \mathfrak{A}_d$. Everything else is evident.

LEMMA 20. *The lattice $\mathcal{L}_{((1,0,1))}$ is not modular.*

The proof is similar to that of Lemma 19.

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