COLLOQUIUM MATHEMATICUM

VOL. XLVII 1982 FASC. 1

ON BISEMILATTICES. I

BY

J. DUDEK (WROCŁAW)

1. Introduction. In [7] Płonka introduced the notion of a quasilattice and in [6] the quasilattice was called a bisemilattice. An algebra $(B, +, \cdot)$ of type $(2, 2)$ is said to be a *bisemilattice* if it satisfies the following axioms:

(1) $x+x = x$, $x\cdot x = x$;

(2) $x+y = y+x$, $x\cdot y = y\cdot x$;

(3) $(x+y)+z = x+(y+z)$, $(x\cdot y)\cdot z = x\cdot (y\cdot z)$

(in the sequel we shall write xy instead of $x\cdot y$).

The class of all algebras of type $(2, 2)$ satisfying (1) and (2) is denoted by $V(+, \cdot)$, and the class of all bisemilattices by $B(+, \cdot)$. Of course, $B(+, \cdot)$ is a subvariety of the variety $V(+, \cdot)$.

An algebra (A, F) is said to be *proper* if all fundamental polynomials are different and every non-nullary $f \in F$ depends on all its variables.

For a given class K of algebras and a given integer $n > 0$ we denote by $N_n(K)$ (or, shortly, N_n) the set of all k for which there exists an algebra $\mathfrak{A}$ from K such that the cardinality $p_n(\mathfrak{A})$ of the set of all essentially n-ary polynomials over $\mathfrak{A}$ is precisely k.

In this paper we present two theorems. In Theorem 1 we establish necessary and sufficient conditions for a bisemilattice to be a lattice, and Theorem 2 deals with the set N_2 for the variety $B(+, \cdot)$, namely we show that the numbers $0, 1, 2, 4, 5$ are in N_2 but $3 \notin N_2$. The equivalence of conditions (i) and (v) in Theorem 1 furnishes an example for the fact that pure set-theoretical assumptions can have algebraical implications.

We shall use here notation and definitions from [2] and [4].

2. Denote by $B_{a.}(+, \cdot)$ the subvariety of the variety $B(+, \cdot)$ which satisfies the additional identity $(x+y)y = y$ (dually, $B_{a+}(+, \cdot)$ denotes the subvariety of $B(+, \cdot)$ of all algebras satisfying $xy+y = y$). Members of $B_{a.}(+, \cdot)$ (or $B_{a+}(+, \cdot)$, respectively) are called *bisemilattices with one absorption law*. Let us mention that there are many interesting subvarieties of the variety $V(+, \cdot)$; namely, the variety of all lattices, the variety of all weak associative lattices (see [1] and [3]), the variety of all distributive quasilattices (see [7]), the variety of all bisemilattices with one distributive law (see [5]).

THEOREM 1. *Let* $(B, +, \cdot) \in B(+, \cdot)$ *and* $\operatorname{card} B \geqslant 2$. *The following conditions are equivalent:*

(i) $(B, +, \cdot)$ *is a lattice;*

(ii) *both polynomials* $(x+y)y$ *and* $xy+y$ *are not essentially binary;*

(iii) $(xy+y)(x+y)$ *is not essentially binary;*

(iv) $(x+y)y+(xy)$ *is not essentially binary;*

(v) $p_2(B, +, \cdot) = 2$.

Before proving this theorem we need some lemmas.

LEMMA 1. *If* $(B, +, \cdot)$ *is proper in* $B(+, \cdot)$, *then* $(x+y)y \neq x$ *and* $xy+y \neq x$.

Proof. If $(x+y)y = x$, then $x = (x+y)y = \big((x+y)+y\big)y = x+y = y+x = y$, a contradiction. Analogously we prove that $xy+y \neq x$.

LEMMA 2. *If* $(B, +, \cdot)$ *is a bisemilattice and* $(x+y)y$ *is commutative, then* $(x+y)y = x+y$ (*the dual version is true for the polynomial* $xy+y$).

Proof. If $(x+y)y = (y+x)x$, then

$$x+y = (x+y)(x+y) = \big(y+(x+y)\big)(x+y) = \big((x+y)+y\big)y = (x+y)y.$$

As above, $xy+y = yx+x$ implies $xy+y = xy$.

Observe that there exists a proper bisemilattice $(B, +, \cdot)$ for which $(x+y)y = x+y$. For example, take the set of all natural numbers for B, the least common multiple $[x, y]$ for $x+y$, and $\max(x, y)$ for xy.

LEMMA 3. *If* $(B, +, \cdot)$ *is a proper algebra from* $B(+, \cdot)$, *then* $(x+y)y$ *and* $xy+y$ *cannot be simultaneously commutative.*

Proof. Indeed, if $(x+y)y$ and $xy+y$ are both commutative then using Lemma 2 we have $x+y = (x+y)y$ and $xy = xy+y$. From the latter identity we get $xy = xy+y = xy+x = xy+y+x$. Hence we obtain

$$x+y = (x+y)x = \big((x+y)y\big)x = (x+y)(xy) = (x+y)(xy+x+y)$$
$$= \big((xy)+(x+y)\big)(x+y) = xy+x+y = xy,$$

which is a contradiction with the assumption $x+y \neq xy$. This completes the proof of the lemma.

LEMMA 4. *If* $(B, +, \cdot)$ *is a proper bisemilattice and* $(x+y)y$ *is commutative, then* $xy+y$ *is essentially binary and non-commutative* (*and, dually, if* $xy+y$ *is commutative, then* $(x+y)y$ *is essentially binary and non-commutative*).

Proof. From Lemmas 1 and 2 we get $xy+y \neq x$ and $(x+y)y = x+y$. If $xy+y = y$, then

$$xy = y(xy) = (xy+y)(xy) = \big(y+(xy)\big)(xy) = y+(xy) = xy+y = y,$$

a contradiction. Thus we have proved that $xy+y$ is essentially binary. To get our assertion we use Lemma 3. Analogously one can prove the dual part of the lemma.

Proof of Theorem 1. Of course, if $(B, +, \cdot)$ is a lattice, then each of the conditions (ii)-(v) is fulfilled.

(ii) $\Rightarrow$ (i) follows immediately from Lemma 1 and from the fact that $x+y$ and xy are idempotent.

(iii) $\Rightarrow$ (i). If $(xy+y)(x+y) = x$, then

$$xy = \big((xy)y+y\big)(xy+y) = (xy+y)(xy+y) = xy+y,$$

and hence

$$x = (xy+y)(x+y) = (xy)(x+y) = (yx)(y+x) = y,$$

a contradiction with $\operatorname{card} B \geqslant 2$. Now, let $y = (xy+y)(x+y)$. Then using the same arguments as above we get $xy+y = y$, and hence

$$y = (xy+y)(x+y) = (x+y)y,$$

which proves that $(B, +, \cdot)$ is a lattice.

(iv) $\Rightarrow$ (i). The proof is similar (dual) to the previous one.

(v) $\Rightarrow$ (i). By the assumption we infer that $x+y$ and xy are the only different essentially binary polynomials over $(B, +, \cdot)$, and hence $(B, +, \cdot)$ is a proper algebra. If the polynomial $(x+y)y$ is essentially binary, then, by the assumption $p_2(B, +, \cdot) = 2$, $(x+y)y$ is commutative, since otherwise $p_2(B, +, \cdot) \geqslant 4$, a contradiction. Now, using Lemmas 2 and 4 we infer that $x+y = (x+y)y$, and the polynomial $xy+y$ is essentially binary and non-commutative, which implies $p_2(B, +, \cdot) \geqslant 4$, a contradiction. Thus we have just proved that $(x+y)y$ is not essentially binary. Using Lemma 1 we get $(x+y)y = y$. Consider now the polynomial $xy+y$. One can prove, as above, that also $xy+y$ is not essentially binary. Hence $(B, +, \cdot)$ satisfies (ii), which proves that $(B, +, \cdot)$ is a lattice. The proof of the theorem is completed.

THEOREM 2. *There are no bisemilattices for which $p_2 = 3$; however, there are bisemilattices for which $p_2 = k$, where $k \in \{0, 1, 2, 4, 5\}$, i.e., $k \in N_2\big(B(+, \cdot)\big)$ for $0 \leqslant k \leqslant 5$ and $k \neq 3$.*

Proof. Of course, every one-element bisemilattice has the property $p_2 = 0$, and hence $0 \in N_2\big(B(+, \cdot)\big)$. Observe also that the two-element semilattice $(\{0, 1\}, \vee)$ can be treated as an algebra $(\{0, 1\}, \vee, \vee)$ of type $(2, 2)$ from the variety $B(+, \cdot)$. It is clear that for this algebra we have

$p_2 = 1$. Therefore $1 \in N_2(B(+,\cdot))$. Using (v) of Theorem 1 we infer that $2 \in N_2(B(+,\cdot))$.

To prove that $4 \in N_2(B(+,\cdot))$, we take any free distributive quasilattice with at least two free generators (see [7]). Using the axioms of a distributive quasilattice (see [7]) and the Marczewski formula of [4] to describe the set $A^{(n)}(\mathfrak{A})$ for a given algebra $\mathfrak{A}$, one can verify that in such algebras the only essentially binary polynomials are $x+y$, xy, $(x+y)y$, and $(y+x)x$. Hence $p_2 = 4$.

Now, we prove that $3 \notin N_2(B(+,\cdot))$. Let $(B,+,\cdot)$ be a bisemilattice for which $p_2 = 3$. Then $(B,+,\cdot)$ is proper, since otherwise one can easily check that $p_2(B,+,\cdot) \leqslant 1$. Consider the polynomials $(x+y)y$ and $xy+y$. By Lemma 1, we have $(x+y)y \neq x$ and $xy+y \neq x$. If $(x+y)y$ is essentially binary, then it must be commutative since $p_2(B,+,\cdot) = 3$. Using Lemma 4 we infer that $xy+y$ is essentially binary and non-commutative, and hence $p_2(B,+,\cdot) \geqslant 4$, a contradiction. Thus it remains to examine the case $(x+y)y = y$. Let us now consider the polynomial $xy+y$. Recall that, by Lemma 1, $xy+y \neq x$. If $xy+y = y$, then $(B,+,\cdot)$ is a proper lattice, and consequently $p_2(B,+,\cdot) = 2$, a contradiction. If $xy+y$ is essentially binary, then, by the assumption $p_2(B,+,\cdot) = 3$, it follows that $xy+y$ is commutative. Now, Lemma 4 implies that the polynomial $(x+y)y$ is essentially binary and non-commutative, which contradicts $(x+y)y = y$. Therefore, we have proved that there exists no bisemilattice for which $p_2 = 3$.

However, we prove that there exists a bisemilattice for which $p_2 = 5$. Moreover, this bisemilattice can be taken from the variety $B_{a.}(+,\cdot)$. Consider any free algebra $\mathscr{F}$ with at least two free generators in a subvariety of the variety $B(+,\cdot)$ defined by the following additional identities:

$$(x+y)y = y, \qquad (xy+y)(x+y) = xy+y, \qquad (xy+y)(yx+x) = xy.$$

Of course, every lattice satisfies the above identities. Using the axioms defining the above variety and the Marczewski formula to describe the set $A^{(n)}(\mathfrak{A})$, we have

$$A^{(2)}(\mathscr{F}) = \{x, y, x+y, xy, x+y+xy, xy+y, yx+x\}.$$

Using again Lemmas 1 and 2 and the fact that $\mathscr{F}$ is free in the considered variety we conclude that $x+y$, xy, $x+y+xy$, $xy+y$, $yx+x$ are the only essentially binary polynomials over $\mathscr{F}$. Hence $p_2(\mathscr{F}) = 5$ and, of course, $\mathscr{F} \in B_{a.}(+,\cdot)$. Thus the proof of the theorem is completed.

Added in proof. We have recently learnt that bisemilattices with $p_2 = 5$ were described by J. Gałuszka in *Bisemilattices with five essentially binary polynomials* (preprint).

REFERENCES

[1] E. Fried, *A generalization of ordered algebras systems*, Acta Scientiarum Mathematicarum (Szeged) 31 (1970), p. 233-244.
[2] G. Grätzer, *Universal algebra*, Van Nostrand, 1979.
[3] — and E. Fried, *Nonassociative extension of the class of distributive lattices*, Pacific Journal of Mathematics 49 (1970), p. 59-78.
[4] E. Marczewski, *Independence and homomorphisms in abstract algebras*, Fundamenta Mathematicae 50 (1961), p. 45-61.
[5] R. McKenzie and A. Romanowska, *Varieties of ·-distributive bisemilattices*, in: *Contributions to general algebra*, Proceedings of the Klagenfurt Conference, Klagenfurt 1979.
[6] R. Padmanabhan, *Regular identities in lattices*, Transactions of the American Mathematical Society 158 (1971), p. 179-188.
[7] J. Płonka, *On distributive quasilattices*, Fundamenta Mathematicae 60 (1967), p. 197-200.

WROCŁAW UNIVERSITY
INSTITUTE OF MATHEMATICS

Reçu par la Rédaction le 23. 1. 1980