COLLOQUIUM MATHEMATICUM

VOL. XV | 1966 | FASC. 2

A PROPERTY OF DIFFERENTIABLE FUNCTIONS

BY

W. WOJTYŃSKI (WARSAW)

Let C^k denote the class of all real valued functions on the real line which have continuous k-th derivative, $k = 0, 1, 2, \ldots, +\infty$. Obviously, $C^\infty = \bigcap_k C^k$.

The letters f, g, ψ, F will denote real-valued functions of one real variable. The symbol $f\circ g$ denotes the composite function, i.e. $f\circ g(x) = f(g(x))$ for all x. f^n denotes the n-th power of the function f and $f^{(k)}$ is its k-th derivative (n and k are positive integers). f^{-1} denotes the inverse of the function f. Symbols φ_h are reserved for translations, i.e. $\varphi_h(x) = \varphi(x+h)$.

The purpose of this paper is to establish the following result:

THEOREM [1]. *If f is a one-to-one mapping of the real line onto itself and $f\circ\psi\circ f^{-1}\in C^k$ for every $\psi\in C^\infty$, then $f\in C^k$.*

The proof follows immediately from Propositions 1 and 2 given below.

PROPOSITION 1. *If f is a one-to-one function from the real line onto itself and $f\circ\psi\circ f^{-1}$ is continuous for any $\psi\in C^\infty$, then f is a homeomorphism.*

Proof. Let $\varepsilon > 0$ and let x_0 be an arbitrary real number. Let us choose ψ in C^∞ such that $\psi(x) = 0$ for $|x-f^{-1}(x_0)| > \varepsilon$, $0 \leqslant \psi \leqslant 1$, and $f^{-1}(x_0) = 1$.

From the continuity of $f\circ\psi\circ f^{-1}$ it follows that for any $\eta > 0$ there exists a $\delta > 0$ such that $|x-x_0| < \delta$ implies $|f\circ\psi\circ f^{-1}(x) - f\circ\psi\circ f^{-1}(x_0)| < \eta$.

Now let $\eta = |f(1)-f(0)|$. Then $|f\circ\psi\circ f^{-1}(x) - f\circ\psi\circ f^{-1}(x_0)| < \eta$ implies $\psi\circ f^{-1}(x) \neq 0$, whence $|f^{-1}(x)-f^{-1}(x_0)| < \varepsilon$.

Thus the function f^{-1} is continuous.

PROPOSITION 2. *Let k be a positive integer and let f be a homeomorphism of the real line onto itself. If both $f\circ\varphi_h\circ f^{-1}$ and $f^{-1}\circ\varphi_h\circ f$ are in C^k for any translation φ_h, then f and f^{-1} belong to C^k.*

---

[1] This theorem substantiates a conjecture of C. Bessaga and A. Pełczyński. In the case of $k = 1$ it has been independently proved by C. Foiaş (oral communication).

The proof consists of three parts.

A. The case $k = 1$. Since f is a homeomorphism, f is monotone. By Lebesgue's theorem, f has a derivative at some points, say at a point x_0.

Let us write $g_h = f \circ \varphi_h \circ f^{-1}$. Since $g_h \in C^1$ for all h, and $g_h \circ f = f \circ \varphi_h$, we conclude that $g_h \circ f$ has a derivative at the point x_0. Thus, for every h, the function $f \circ \varphi_h(x) = f(x+h)$ has a derivative at the point x_0. This implies that $f(x)$ has a derivative for all x.

The function $f^{(1)}$, being of Baire's first class, has a point of continuity, say x_0. The function $(g_h \circ f)^{(1)} = (f \circ \varphi_h)^{(1)} = f^{(1)} \circ \varphi_h$ is then continuous at the point x_0 for all h, but this means that $f^{(1)}$ is continuous.

B. The case $k = 2$. We shall need the following

LEMMA. *Let F be a continuous function. If the difference $F \circ \varphi_h - F$ is of the class C^1 for every translation φ_h, then F is of the class C^1.*

Proof [2]. The proof is based on the following result (cf. Young [4], Denjoy [2], Saks [3], and Banach [1]):

(+) *If g is a real-variable function, then*

$$A_g = \left\{ x:\ -\infty < \underline{\lim}_{h \to 0} \frac{g(x+h)-g(x)}{h} < \overline{\lim}_{h \to 0} \frac{g(x+h)-g(x)}{h} < +\infty \right\}$$

is of Lebesgue's measure zero.

(++) From the assumption of the lemma it follows that if at one point the upper (lower) right (left) side derivative of F is equal to $\pm\infty$, then at every point this derivative is equal to $\pm\infty$. It follows also that if the function F has the derivative at one point, then it has the derivative everywhere. It is also obvious that A_F is empty.

Let us suppose that F is not differentiable. Since every monotone function has a point of differentiability, F is not monotone on any interval. Thus we can choose points $x_1 < x_2 < x_3 < x_4$ such that $F(x_1) < F(x_2)$ and $F(x_3) > F(x_4)$. Function F, being continuous, attains its least upper bound in the interval $[x_1, x_4]$, and attains it in the interior of the interval $[x_1, x_4]$, say at the point x_0. At the point x_0 there is then a local maximum of F, and therefore the upper right side derivative is not equal to $+\infty$ and the lower left side derivative is not equal to $-\infty$. Hence it follows from (++) that such inequalities hold at any point. In the same way we can get a point y_0 at which the function F attains its local minimum. Then at y_0 the lower right side derivative is not equal to $-\infty$, and the upper left side derivative is not equal to $+\infty$. As before, such inequalities hold for any point. Consequently,

---

(2) This proof has been comunicated to the author by Mr. S. Kwapień.

the function F has both upper and lower derivatives finite at any point, which contradicts the quoted theorem $(+)$.

Function $F^{(1)}$ being of Baire's first class has points of continuity. Reasoning in an analogous way to the part A of our proof, we infer that $F^{(1)}$ is continuous.

Now, let us come back to the proof of Proposition 2. Suppose that f and f^{-1} are in C^1 and that $f\circ\varphi_h\circ f^{-1}\in C^2$ for $-\infty < h < +\infty$. Thus $(f\circ\varphi_h\circ f^{-1})^{(1)}$ and $(f\circ\varphi_h\circ f^{-1})^{(1)}\circ f$ are in C^1. A simple computation gives:

$$(1)\qquad (f\circ\varphi_h\circ f^{-1})^{(1)} = \frac{f^{(1)}\circ\varphi_h\circ f^{-1}}{f^{(1)}\circ f^{-1}}.$$

Thus

$$(2)\qquad (f\circ\varphi_h\circ f^{-1})^{(1)}\circ f = \frac{f^{(1)}\circ\varphi_h}{f^{(1)}}\in C^1.$$

Since f and f^{-1} belong to $C^1, f^{(1)}(x)\neq 0$ for $-\infty < x < +\infty$. And since $f^{(1)}$ is continuous, we can assume without loss of generality that $f^{(1)}(x) > 0$ for $-\infty < x < +\infty$. Thus $F = \log\circ f^{(1)}\in C^1$, and (2) implies:

$$F\circ\varphi_h - F\in C^1 \quad \text{for} \quad -\infty < h < +\infty.$$

Thus the lemma implies $F = \log\circ f^{(1)}\in C^1$. Hence $f^{(1)} = e^F\in C^1$, and therefore $f\in C^2$.

C. The case $k > 2$. Now we assume that Proposition 2 holds for $j\leqslant n$ $(n\geqslant 2)$ and let $f\circ\varphi_h\circ f^{-1}\in C^{n+1}$ for all translations φ_h. It follows that f and f^{-1} are in C^n.

First let us observe that *if* $f\in C^n$ *and* $g\in C^{n+1}$, *then*

$$(3)\qquad (g\circ f)^{(j)} = (g^{(1)}\circ f)\cdot f^{(j)} + \psi_j,$$

where ψ_j *belongs to* C^{n-j+1} *for* $j = 1, 2, \ldots, n$.

This is obvious for $j = 1$. Assuming that (3) holds for some $j\leqslant n-1$ and differentiating both parts of the formula, we verify (3) for $j+1$.

Now, let $g = f\circ\varphi_h\circ f^{-1}$. We have $g\circ f = f\circ\varphi_h$. Applying (3) for $j = n$, we infer that

$$(f\circ\varphi_h)^{(n)} - (g^{(1)}\circ f)\cdot f^{(n)}\in C^1.$$

Replacing $g^{(1)}$ by the right-hand side of (1) we get

$$(f\circ\varphi_h)^{(n)} - \left[\left(\frac{f^{(1)}\circ\varphi_h\circ f^{-1}}{f^{(1)}\circ f^{-1}}\right)\circ f\right]\cdot f^{(n)} = (f\circ\varphi_h)^{(n)} - \frac{f^{(1)}\circ\varphi_h}{f^{(1)}}\cdot f^{(n)}\in C^1.$$

Since $n\geqslant 2$, $f^{(1)}$ and so $f^{(1)}\circ\varphi_h$ are in C^1. Thus

$$\frac{(f\circ\varphi_h)^{(n)}}{f^{(1)}\circ\varphi_h} - \frac{f^{(n)}}{f^{(1)}}\in C^1.$$

Since $f^{(n)}/f^{(1)}$ is continuous and $(f\circ\varphi_h)^{(n)}/f^{(1)}\circ\varphi_h = (f^{(n)}/f^{(1)})\circ\varphi_h$, to complete the proof it is enough to apply the Lemma with $F = f^{(n)}/f^{(1)}$.

Added in proof. An alternative formulation of the main result: *Let $G = \{f_h\}$ be a one-parameter transitive group of C^k-diffeomorphisms of the real line R. If G is continuous with respect to the parameter h and G acts in a free way (i. e. $f_h x = x$ iff $h = 0$), then G is of the form $f_h(x) = F\circ\varphi_h\circ F^{-1}$, where F is a 'ixed C^k-diffeomorphism.*

In fact, let $f_h(x)$ be diffeomorphisms for $-\infty < h < +\infty$. Write $F(h) = f_h(0)$. Then $h_1 \neq h_2$ implies $F(h_1) \neq F(h_2)$. Indeed, if we had $F(h_1) = F(h_2)$, then we would get $0 = F(h_1) - F(h_2) = f_{h_1}(0) - f_{h_2}(0) = f_{h_2}\circ f_{h_1-h_2}(0) - f_{h_2}(0)$, i. e. $f_{h_2}(f_{h_1-h_2}(0)) = f_{h_2}(0)$, whence $f_{h_1-h_2}(0) = 0$, which contradicts the assumption that the group G acts in a free way.

From the transitivity of the group G it follows that F is onto. Since F is one-to-one and continuous, F is a homeomorphism of R onto R. The assertion follows then from Proposition 2 and the following equalities:

$$F^{-1}(x) = \{t : f_t(0) = x\}, \qquad \varphi_h\circ F^{-1}(x) = t+h,$$

$$F\circ\varphi_h\circ F^{-1}(x) = F(t+h) = f_{t+h}(0) = f_h(f_t(0)) = f_h(x).$$

REFERENCES

[1] S. Banach, *Sur les ensembles de points ou la dérivée est infinie*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences (Paris) 173 (1921), p. 457-459.

[2] A. Denjoy, *Mémoire sur les nombres dérivés des fonctions continues*, Journal de Mathématiques Pures et Appliquées (Paris) 7 (1915), p. 105-240.

[3] S. Saks, *Sur les nombres dérivés de fonctions*, Fundamenta Mathematicae 5 (1924), p. 98-104.

[4] W. H. Young, *La symétrie de structure des fonctions de variables réelles*, Bulletin de Sciences Mathématiques (Paris) 52 (1928), p. 265-280.

Reçu par la Rédaction le 5. 6. 1965