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CONCERNING SOME THEOREMS OF MARCZEWSKI ON ALGEBRAIC INDEPENDENCE

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In [3] Marczewski has stated an "exchange theorem" for general algebraic systems with finitary operations which enabled him to prove his result on the cardinal numbers of bases of such system constituting an arithmetical progression. This exchange theorem deals with families of two subsets of the algebraic system and is of unsymmetric character. In this short note, it is first shown that this exchange theorem is a special case of a much more general theorem of symmetric character dealing with arbitrary finite or infinite families of subsets; it even holds true in algebraic systems with arbitrary finitary or infinitary operations:

THEOREM 1. *Let $(M_t)_{t\in T}$ and $(N_t)_{t\in T}$ be families of subsets of an algebra A such that:*

1. *the subsets M_t are pairwise disjoint;*
2. $\mathrm{C}M_t = \mathrm{C}N_t$ *for all $t\in T$ (i. e. M_t and N_t generate the same subalgebra of A);*
3. $\bigcup\limits_{t\in T} M_t$ *is a B-independent subset of algebra A, for a certain algebra B (i. e. each mapping $\alpha: \bigcup M_t \to B$ can be uniquely extended to a homomorphism $\varphi: \mathrm{C}\bigcup M_t \to B$)* (1);
4. *the subsets N_t are B-independent subsets of A.*

Then $\bigcup\limits_{t\in T} N_t$ is a B-independent subset of A.

Proof. Let β be a B-valuation of $\bigcup N_t$. Since the N_t are B-independent, for each $t\in T$ there exists a unique homomorphism $\varphi_t: \mathrm{C}N_t \to$ $\to B$ extending the restriction of β to N_t. Since $\bigcup M_t$ is B-independent and the M_t are pairwise disjoint, there exists a homomorphism $\varphi: \mathrm{C}\bigcup M_t \to B$, extending the restrictions of the φ_t to $M_t \subseteq \mathrm{C}N_t$ and

(1) Following the ideas of Marczewski [3], the general theory of this notion has been developped in [4] and [5].

therefore, since $CM_t = CN_t$, extending the φ_t themselves; thus φ equals φ_t, i. e. β, on $N_t : \varphi$ is an extension of β. But $C\bigcup M_t = C\bigcup N_t$, completing the proof.

Marczewski's "exchange theorem" [3], 2.4 (ii), is the special case $T = \{1, 2\}$, $M_2 = N_2$, $B = A$: assuming $M_1 \cup M_2$ to be an (A-) independent subset of A, one concludes $N_1 \cup M_2$ to be (A-) independent too.

There is the closely related (²)

THEOREM 2. *Let $(M_t)_{t\in T}$ be a family of pairwise disjoint subsets of algebra A, $(N_t)_{t\in T}$ a family of pointwise disjoint subsets of algebra B. Moreover, let $\bigcup M_t$ be a B-independent generating subset of A, $\bigcup N_t$ an A-independent generating subset of B. If, for each $t \in T$, subalgebra $CM_t \subseteq A$ is isomorphic to subalgebra $CN_t \subseteq B$, then algebras A and B themselves are isomorphic.*

Proof. By the axiom of choice, for each $t\in T$ there exists an isomorphism $\varphi_t : CM_t \to CN_t$. Since the set $\bigcup M_t$ is B-independent and the sets M_t are pairwise disjoint, there exists a homomorphism $\varphi : C\bigcup M_t = A \to B$ that equals φ_t on M_t and therefore on CM_t. In the same way, we obtain a homomorphism $\psi : C\bigcup N_t = B \to A$ that equals the converse isomorphism $\psi_t = \varphi_t^{-1} : CN_t \to CM_t$ on N_t and therefore on CN_t. Then $\psi \circ \varphi$ is an endomorphism of A that equals the identity on the generating subset $\bigcup M_t$ and therefore is the identity of A; in the same way, $\varphi \circ \psi$ is the identity of B: $\varphi : A \to B$ and $\psi : B \to A$ are converse isomorphisms.

A result stronger than Marczewski's quoted theorem of the arithmetical progression may be obtained by making use of theorem 2 instead of theorem 1. Let us consider a primitive class of algebras, i. e. a class $\mathfrak{A}$ which is closed with respect to direct products, subalgebras, and homomorphic images. As is well-known, this is the same as $\mathfrak{A}$ to be equationally definable in the sense of Birkhoff and Tarski. Moreover, let $\mathfrak{A}$ be non-trivial, i. e. $\mathfrak{A}$ shall contain an algebra B of cardinality $|B| \geqslant 2$. As is well-known, for any cardinal number $\mathfrak{m}$ there is one and up to isomorphism only one algebra $A_{\mathfrak{m}} \in \mathfrak{A}$ generated by a subset M of cardinality $|M| = \mathfrak{m}$ such that M is an $\mathfrak{A}$-free ($\mathfrak{A}$-independent), i. e. A-independent for each $A \in \mathfrak{A}$, subset of $A_{\mathfrak{m}}$ (³).

In order to become more concrete, let us remark that a class $\mathfrak{A}$ is primitive if and only if there is an algebra A such that $\mathfrak{A}$ precisely consists of the homomorphic images of the subalgebras of the direct powers of A,

$$\mathfrak{A} = \mathscr{H}\Big(\mathscr{S}\big(\mathscr{P}(A)\big)\Big);$$

(²) Cf. Marczewski [3], 2.4 (i).

(³) In [8], the indispensable hypothesis of the primitivity and non-triviality of class $\mathfrak{A}$ has not been stated explicitly.

then $\mathfrak{A}$ is non-trivial if and only if $|A| \geqslant 2$ (4). Then the $\mathfrak{A}$-free algebra $A_{\mathfrak{m}}$ can be represented as the algebra $H^M(A) \subseteq A^{A^M}$ of all algebraic operations on A of type M where $|M| = \mathfrak{m}$: the set $E^M(A)$ of the identity operations on A of type M is an A- and therefore $\mathfrak{A}$-independent generating subset of $H^M(A)$, of cardinality $|E^M(A)| = |M| = \mathfrak{m}$.

Now, as a consequence of theorem 2, we have

THEOREM 3. *Let $(\mathfrak{m}_t)_{t \in T}$ and $(\mathfrak{n}_t)_{t \in T}$ be two families of cardinal numbers. If, for each $t \in T$, the algebras $A_{\mathfrak{m}_t}$ and $A_{\mathfrak{n}_t}$ are isomorphic, then so are algebras $A_{\mathfrak{m}}$ and $A_{\mathfrak{n}}$ where $\mathfrak{m} = \sum \mathfrak{m}_t$, $\mathfrak{n} = \sum \mathfrak{n}_t$.*

Proof. $A_{\mathfrak{m}}$ is $\mathfrak{A}$-freely generated by the union $\bigcup M_t$ of a family of pairwise disjoint sets M_t of cardinality $|M_t| = \mathfrak{m}_t$; $A_{\mathfrak{n}}$ is $\mathfrak{A}$-freely generated by the union $\bigcup N_t$ of a family of pairwise disjoint sets N_t of cardinality $|N_t| = \mathfrak{n}_t$. Then M_t and N_t are $\mathfrak{A}$-free subsets of algebra $A_{\mathfrak{m}}$ or $A_{\mathfrak{n}}$ respectively: subalgebra $CM_t \subseteq A_{\mathfrak{m}}$ is an algebra $A_{\mathfrak{m}_t}$, subalgebra $CN_t \subseteq A_{\mathfrak{n}}$ an algebra $A_{\mathfrak{n}_t}$; by hypothesis, CM_t and CN_t are isomorphic, by theorem 2, algebras $A_{\mathfrak{m}}$ and $A_{\mathfrak{n}}$ are isomorphic.

Algebraically speaking, theorem 3 states that the equivalence relation on cardinal numbers associated with class $\mathfrak{A}$ by the definition

$$\mathfrak{m} \underset{\mathfrak{A}}{\equiv} \mathfrak{n} \text{ if and only if } A_{\mathfrak{m}} \cong A_{\mathfrak{n}}$$

is totally additive. In particular, when restricted to natural numbers, it is a congruence relation of their additive semigroup: this is — for the special case of algebras with finitary operations — essentially the statement of Świerczkowski [8], corollary 1 of lemma 1, which, due to the above remark on primitive classes, is equivalent with the statement of Goetz and Ryll-Nardzewski [1] theorem 3 (5).

COROLLARY 1. *The cardinal numbers $\mathfrak{n}$ such that $A_{\mathfrak{m}} \cong A_{\mathfrak{n}}$ constitute a complete residue class of a totally additive equivalence relation on cardinal numbers.*

Let us note that the residue classes of congruence relations of the additive semigroup of natural numbers are precisely the arithmetical progressions

$$A_{kd} = \{k + nd \mid n = 0, 1, 2, \ldots\} \quad (k, d = 0, 1, 2, \ldots) \text{ (6)}.$$

So we have

COROLLARY 2. *The natural numbers n such that $A_m \cong A_n$ constitute an arithmetical progression.*

For algebras with finitary operations, this is theorem 1 of Świerczkowski [8]. An essentially equivalent formulation of corollaries 1 and 2:

(4) In [1], this assumption $|A| \geqslant 2$ is tacitly left to the reader.

(5) Cf. footnotes (3) and (4). The technique of proof in [8] and [1] makes use of algebraic operations instead of homomorphisms.

(6) Cf. [6], corollary of theorem 2.

COROLLARY 3. *Let A be an arbitrary algebra such that $|A| \geqslant 2$. Then the powers $|M|$ of bases (= A-independent generating subsets) M of A constitute a complete residue class of a totally additive equivalence relation on cardinal numbers, the finite ones among them an arithmetical progression.*

For as is well-known, algebra A, of cardinal number $|A| \geqslant 2$, contains a basis of cardinal number $\mathfrak{m}$ if and only if A is isomorphic to $H^M(A)$ where $|M| = \mathfrak{m}$. In the special case of algebras with finitary operations, the last part of this corollary is Marczewski's theorem of the arithmetical progression, [3], 2.4 (iv), which has also been proved by Goetz and Ryll-Nardzewski [1], theorem 5 (7).

The residue classes of cardinal numbers associated with classes of algebras are not completely arbitrary. For instance, in a class of algebras with finitary operations, each infinite cardinal number forms a residue class for its own; this is an immediate consequence of a property of the associated closure operator C (8). A further restriction on the residue classes which holds true for algebras of arbitrary finitary or infinitary type:

THEOREM 4. *A_0 is not isomorphic to any of the algebras $A_{\mathfrak{m}}$ such that $\mathfrak{m} \geqslant 1$.*

Proof. Let $A_{\mathfrak{m}}$ be isomorphic to A_0; we want to show that $\mathfrak{m} = 0$. $A_{\mathfrak{m}}$ is $\mathfrak{A}$-freely generated by a subset M of cardinal number $|M| = \mathfrak{m}$. Then the empty set $\emptyset$, being a subset of M, also is an $\mathfrak{A}$-independent subset of $A_{\mathfrak{m}}$, hence subalgebra $C\emptyset$ (9) is an algebra A_0. By assumption, there exists an isomorphism $\varphi : C\emptyset \to A_{\mathfrak{m}}$. But φ equals the identity of $A_{\mathfrak{m}}$ on subset $\emptyset$ and therefore on subalgebra $C\emptyset$, hence $C\emptyset = A_{\mathfrak{m}}$: $\emptyset$ is a generating subset of $A_{\mathfrak{m}}$. Assuming $\mathfrak{m} \neq 0$ and therefore $M \neq \emptyset$, we find an element $a \in M$. Let B be an arbitrary algebra belonging to class $\mathfrak{A}$, let b_1, b_2 be arbitrary elements of B. M being B-independent, there are homomorphisms $\varphi_1, \varphi_2 : CM = A_{\mathfrak{m}} \to B$ such that $\varphi_1(a) = b_1$, $\varphi_2(a) = b_2$. Yet both homomorphisms are equal on the generating subset $\emptyset$ of $A_{\mathfrak{m}}$, hence on the whole of $A_{\mathfrak{m}}$, in particular $\varphi_1(a) = \varphi_2(a)$, i. e. $b_1 = b_2$: we obtain $|B| \leqslant 1$, contradicting hypothesis of the non-triviality of class $\mathfrak{A}$.

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(7) Marczewski's proof uses theorem 1 instead of theorem 2; Goetz and Ryll-Nardzewski use algebraic operations instead of homomorphisms. An earlier special case of the theorem of the arithmetical progression has been found by Świerczkowski [7]; cf. also Jónsson and Tarski [2].

(8) Cf. Marczewski [3], 1.3 (iv).

(9) Here $C\emptyset$ has the meaning given to it in [4] and [5], which is slightly different from the use in [1], [3], [7], and [8].

Świerczkowski [8] theorem 2 has proved the remarkable result that any arithmetical progression not containing 0 is a residue class associated with a class of algebras with finitary operations, thus completely solving the problem of characterization of those residue classes associated with classes of algebras with finitary operations. There remains the problem of generalization of this result to classes of algebras with arbitrary finitary or infinitary operations. This is a part of the following deeper problem: to characterize the totally additive equivalence relations on cardinal numbers which may be considered as associated with suitable classes of algebras (**P 485**).

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