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EPIMORPHISMS AND AMALGAMS

BY

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1. Introduction. The purpose of this essay* is to highlight the connections and interplay between the concepts of epimorphisms, dominions and amalgams. This will be done largely within the context of algebraic semigroup theory, although the definitions and, indeed, even some of the theorems presented are equally at home in other algebras and other categories.

The concept of epimorphism is the element-free analogue of the notion of surjective mapping. Let C be a category. A morphism α of C is an *epimorphism* (epi, for short) if whenever β and γ are morphisms of C such that $\alpha\beta = \alpha\gamma$, then $\beta = \gamma$. An epimorphism is thus a pre-cancellative morphism (or left cancellative if we compose maps, as I shall here, from left to right). The dual concept of *monomorphism* applies to post-cancellative morphisms and will not be discussed in this paper.

If C is a concrete category (meaning that its objects are sets, perhaps with structure, and the morphisms are functions of these underlying sets), it is obvious that any surjective morphism is epi. Indeed, to say that $\alpha: A \to B$ is epi is the same as saying that whenever $\beta, \gamma: B \to C$ are such that $\beta | A\alpha = \gamma | A\alpha$, then $\beta = \gamma$. If this is the case, we say that $A\alpha$ is *dense* in B or that $A\alpha$ is *epimorphically embedded* in B via the inclusion $i: A\alpha \to B$. Observe that the composition of epimorphisms is itself an epimorphism, while, as a partial converse, if $\alpha\beta$ is epi, then β is epi. It follows from this that $\alpha: A \to B$ is epi if and only if the inclusion $i: A\alpha \to B$ is epi.

We then have a question: for a given category C, are the epimorphisms just the surmorphisms, and if not, can the epis be characterized effectively? This question, which may seem artificial, has led to an astonishing amount of interesting mathematics.

In the category of Sets, with morphisms the arbitrary set maps, the epis are certainly just the surjections: for suppose $\alpha: A \to B$ is not onto, take

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$\beta, \gamma: B \to B$ with β the identity map, and γ a map which is not the identity, but whose restriction to $A\alpha$ agrees with β. Then $\alpha\beta = \alpha\gamma$ but $\beta \neq \gamma$.

Epis are onto in the category of all Topological Spaces and continuous maps. The argument is essentially the same as the Sets case. However, this time we take $\beta, \gamma: B \to C$, where $C = B$ as a set, and C is endowed with the indiscrete topology (C and $\emptyset$ are the only open sets) to ensure that the maps β and γ are continuous, and hence are legitimate morphisms in the category under consideration. Similarly, in the category Ab of all abelian groups and abelian group homomorphisms, it is easy to show that if $\alpha: A \to B$ is not onto, then it is not epi. Take $\beta, \gamma: B \to B/A\alpha$ to be the canonical and zero homomorphisms, respectively. Once again $\alpha\beta = \alpha\gamma$ but $\beta \neq \gamma$.

In the category of Groups and their homomorphisms it is also the case that epis are onto, although the argument used for Ab clearly fails, as in general it is impossible to create the quotient group $B/A\alpha$, since $A\alpha$ may not be a normal subgroup of B. There is nevertheless a fairly short elementary proof of this fact due to Linderholm [25], and [2] contains a proof of similar theme, which shows that epis are onto in the category of all finite groups. However, a group theorist might well answer both questions by stating that this is an immediate corollary to Schreier's Theorem that every (finite) group amalgam can be strongly embedded in a (finite) group. Of this we shall have more to say later.

Incidentally, using the fact that epis are onto for Groups, it is an easy exercise to show that the same is true for the category of all Topological Groups, with morphisms the continuous homomorphisms (again the indiscrete topology saves the day).

These examples may give the impression that there is no difference between epimorphisms and surmorphisms in any category of interest. Let me hasten to dispel this with several instances to the contrary.

Indeed, it is also the case that epis are onto in the category of all T_1 Topological Spaces (see [2] which is the source of most of these introductory examples). But in the category of all Hausdorff Topological Spaces, $\alpha: A \to B$ is epi if and only if $\overline{A\alpha} = B$, i.e. if and only if the image of A under α is dense in B. The proof of the necessity of this condition is in [2] and consists of constructing two distinct continuous maps β and γ which have domain B and which agree on $\overline{A\alpha}$ (which is assumed to be not all of B) whose codomain is another T_2-space formed from two copies of the space B, amalgamating $\overline{A\alpha}$. However, I only prove the sufficiency in detail here as it furnishes our first case of non-surjective epis. Suppose $\overline{A\alpha} = B$ and $\beta, \gamma: B \to C$ are a pair of continuous maps into a T_2-space C, which agree on $A\alpha$. The result follows from the general observation that a continuous mapping whose codomain is a T_2-space is determined by its action on a dense set. To see this, suppose that $b \in B$ is such that $b\beta \neq b\gamma$. Take disjoint open neighbourhoods U and V

about $b\beta$ and $b\gamma$. Then $U\beta^{-1} \cap V\gamma^{-1}$ is open in B, and since it contains at least b, is not empty. Hence there exists $x \in U\beta^{-1} \cap V\gamma^{-1} \cap A\alpha$. But then $x\beta = x\gamma$ is simultaneously a member of both U and V, a contradiction. Therefore $\beta | A\alpha = \gamma | A\alpha$ implies $\beta = \gamma$ if $\overline{A\alpha} = B$.

The category of all Torsion-Free Abelian Groups has a peculiar characterization of its epimorphisms: $\alpha: A \to B$ is epi if and only if $B/A\alpha$ is a torsion group. For example, this states that the inclusion $i: \mathbf{Z}(+) \to \mathbf{Q}(+)$ is an epi in this category, as clearly $\mathbf{Q}/\mathbf{Z}$ is torsion ($n(m/n + \mathbf{Z}) = \mathbf{Z}$). However, this same mapping is not epi in Ab, as it is not onto. Of course, we cannot use that Ab proof to show that i is not epi in the category of Torsion-Free Abelian Groups, as the object $\mathbf{Q}/\mathbf{Z}$ utilized in the argument is not a member of the category.

To prove the assertion, suppose first that $B/A\alpha$ is torsion. Let $b \in B$ and $\beta, \gamma: B \to C$ be two morphisms whose restrictions to $A\alpha$ agree. Since $B/A\alpha$ is torsion, there exists $n \in \mathbf{Z}^+$ such that $nb \in A\alpha$, whence

$$(nb)\beta = (nb)\gamma \Rightarrow n(b\beta - b\gamma) = 0 \Rightarrow b\beta = b\gamma,$$

as C is torsion-free. Hence α is epi if $B/A\alpha$ is torsion.

Conversely, suppose that $B' = B/A\alpha$ is not torsion. Let T be the torsion subgroup of B' so that B'/T is a non-trivial torsion-free group.

Let $k: B \to B'$ and $k_1: B' \to B'/T$ be the canonical homomorphisms, and $k_2: B' \to B'/T$ be the zero homomorphism. Put $\beta = kk_1$ and $\gamma = kk_2$. Then $\beta \neq \gamma$, but $\alpha\beta = \alpha\gamma = 0$.

The categories of Rings and Semigroups are outstanding examples of categories with non-surjective epimorphisms. I say outstanding, as in each case there is a so-called "Zigzag Theorem" which, among other things, gives very useful necessary and sufficient conditions for deciding whether or not a given morphism is epi. Examples of non-surjective epimorphisms in each category are common.

Let us examine the inclusion $i: \mathbf{Z}(+, \cdot) \to \mathbf{Q}(+, \cdot)$. We show that this is an epimorphism in the category of all Rings and ring homomorphisms by proving that any pair of ring homomorphisms $\beta, \gamma: \mathbf{Q} \to \mathbf{R}$ which agree on $\mathbf{Z}$ are in fact equal. We shall show that $(1/n)\beta = (1/n)\gamma$ for all $n \in \mathbf{Z} \setminus \{0\}$, from which it follows easily that $(m/n)\beta = (m/n)\gamma$ for all $m/n \in \mathbf{Q}$. Now

$$\begin{aligned}\frac{1}{n}\beta &= \left(\frac{1}{n}\cdot n \cdot \frac{1}{n}\right)\beta = \left(\frac{1}{n}\right)\beta \cdot \left(n \cdot \frac{1}{n}\right)\beta = \left(\frac{1}{n}\right)\beta\left(n \cdot \frac{1}{n}\right)\gamma \\ &= \left(\frac{1}{n}\right)\beta \cdot n\gamma \cdot \left(\frac{1}{n}\right)\gamma = \left(\frac{1}{n}\right)\beta \cdot n\beta \cdot \left(\frac{1}{n}\right)\gamma = \left(\frac{1}{n}\cdot n\right)\beta\left(\frac{1}{n}\right)\gamma \\ &= \left(\frac{1}{n}\cdot n\right)\gamma\left(\frac{1}{n}\right)\gamma = (1)\gamma \cdot \left(\frac{1}{n}\right)\gamma = \frac{1}{n}\gamma,\end{aligned}$$

as required.

The preceding example can be regarded as an epimorphic embedding in the category of Semigroups by considering $\boldsymbol{Q}$ and $\boldsymbol{Z}$ as sets with purely multiplicative structure. The inclusion i: $(0, 1] \to (0, \infty)$, regarding both intervals as multiplicative semigroups, is also an epimorphism [4]. A similar argument to the previous example establishes this. However, here is an example involving a creature of pure algebraic semigroup theory, the full transformation semigroup [13]. Let S denote the semigroup of all self-maps on some infinite set X under function composition. For $u \in S$ denote the range of u by ∇u. Let U be the subsemigroup of S consisting of the identity map 1 together with $\{u \in S\colon |X \setminus \nabla u| = \infty\}$. Clearly, U is a proper subsemigroup of S. It is also epimorphically embedded. To see this let $d \in S \setminus U$, so that ∇d has a finite complement in X. Take $u \in U$ to be any injection apart from 1. Define $y \in S$ by

$$ny = \begin{cases} nu^{-1} & \text{if } n \in \nabla u, \\ p & \text{otherwise,} \end{cases}$$

where p is a fixed member of X. Observe that $uy = 1$. Now take α, β: $S \to T$ to be any pair of homomorphisms from S to a semigroup T which agree on U. Note that $du \in U$ because $|\nabla du| \leqslant |\nabla u|$, whence $|X \setminus \nabla du| \geqslant |X \setminus \nabla u|$, and the latter is infinite. Hence $(du)\alpha = (du)\beta$, and thus we may write

$$\begin{aligned} d\alpha &= (d \cdot 1)\alpha = d\alpha 1\alpha = d\alpha 1\beta = d\alpha(uy)\beta = d\alpha u\beta y\beta \\ &= d\alpha u\alpha y\beta = (du)\alpha y\beta = (du)\beta y\beta = (duy)\beta = (d \cdot 1)\beta = d\beta, \end{aligned}$$

as required.

The very similar manipulations used in both our examples may at first sight appear ad hoc, but are in fact typical instances of "zigzag" manipulations in both categories.

2. Zigzags and amalgams. In what follows U will denote a subsemigroup of a semigroup S.

Suppose that the inclusion i: $U \to S$ is epi, that is, U is dense in S. We may think of U as a "large" or "dominating" part of S in the sense that the action of any morphism from S is determined by its action on U.

However, in general it may be possible that U "dominates" some, but not all, elements of S. Isbell [18] made this precise by defining the *dominion* of U in S, denoted by $\mathrm{Dom}(U, S)$, to consist of all the elements $d \in S$ which are *dominated* by U in the sense that whenever α, β: $S \to T$ are morphisms such that $\alpha|U = \beta|U$, then $d\alpha = d\beta$. It is easy to see that $\mathrm{Dom}(U, S)$ is a subsemigroup of S containing U. Indeed, it is routine to check that $\mathrm{Dom}(\cdot, S)$ is a closure operator on the subsemigroups of S in the sense that $U \subseteq \mathrm{Dom}(U, S)$; if $U \subseteq V \subseteq S$, then

$$\mathrm{Dom}(U, S) \subseteq \mathrm{Dom}(V, S) \quad \text{and} \quad \mathrm{Dom}(\mathrm{Dom}(U, S), S) = \mathrm{Dom}(U, S).$$

We call a semigroup U *closed* in S if $\mathrm{Dom}(U, S) = U$, and U is *absolutely closed* if it is closed in every containing semigroup S. At the other extreme, U is *dense* in S, or *epimorphically embedded* in S, if $\mathrm{Dom}(U, S) = S$. A weaker condition on U than that of being absolutely closed is that of being *saturated*, which means that U cannot be properly epimorphically embedded in another semigroup.

All these definitions, of course, can be applied equally well to other algebras.

Two points to note: first, it follows from these definitions that to say that every epi *from* a semigroup U is onto is equivalent to saying that every morphic image of U is saturated (there exists a saturated semigroup with a morphic image which is not! (see [12]); also it is *not* always true that a subsemigroup U of S is dense in its own dominion, although it is inconvenient to give a counterexample here (see [13])).

We pause a moment from this barrage of definitions to give a somewhat imprecise introduction to the concept of an amalgam.

A (semigroup) amalgam can be thought of as an indexed family $\{S_i;\ i \in I\}$ of semigroups intersecting in a common subsemigroup U. In this essay we shall only need to consider amalgams of the semigroups S and T with a common *core* subsemigroup U, which we denote by $[S, T; U]$. The amalgam $[S, T; U]$ is then a partial semigroup: some products are meaningful, and $(xy)z = x(yz)$ provided both sides are defined. A natural general question is whether or not the amalgam $[S, T; U]$ can be embedded in another semigroup W (i.e., $[S, T; U] \subseteq W$ and previously defined products in the amalgam are unaltered). In 1927 Schreier showed that any group amalgam is group embeddable, but this simple answer does not suffice for semigroups.

I should mention that a proper definition of amalgam consists of *disjoint* semigroups, equipped with monomorphisms Φ_i that embed the core U into each of the S_i. To then say that the amalgam is embedded in W means there exist monos $\lambda\colon U \to W$, $\lambda_i\colon S_i \to W$ such that $\Phi_i \lambda_i = \lambda$ for all i and $S_i \lambda_i \cap S_j \lambda_j = U\lambda$ for all $i \neq j$. However, in the description of the theory which follows, I shall speak as if all these monos were inclusions, and not distinguish between the domain and range of each of these inclusions. In other words, we think of amalgams as introduced above.

A natural candidate for a semigroup W into which the amalgam $[S, T; U]$ might be embedded is the so-called *amalgamated free product* of S and T over U. This is constructed via the free product of S and T. To introduce the latter concept it is best to think of S and T as disjoint semigroups each containing a copy U_1 and U_2, respectively, of U. We use the notation u_1, v_1, etc. [u_2, v_2, etc.] to denote typical members of the copy U_1 [U_2].

The free product of S and T, denoted by $S * T$ consists of finite sequences, or *words*, whose letters alternately come from S and T. The product of two members w_1, w_2 of $S * T$ is defined by concatenation if the last letter of w_1 and the first of w_2 do not come from the same semigroup, otherwise $w_1 w_2$ is defined by first forming the concatenated word, and then by performing the multiplication of the adjacent end letters in the semigroup (S or T) of which they are both members. The amalgamated free product $S *_U T$ of $[S, T; U]$ is formed by partitioning $S * T$ into equivalence classes: w_1 is in the same class as w_2 if each word can be transformed into the other by a finite sequence of transitions in which some letters from U_1 are replaced by their counterparts in U_2 and vice versa (this allows the members of U to, in effect, act equally as members of S or T). These equivalence classes can now be properly multiplied by means of their representatives (i.e. this partition is a congruence on $S * T$, and $S *_U T$ is a morphic image of $S * T$).

Given two members of $S * T$ it will in general be very difficult to decide whether or not they represent the same member of $S *_U T$. Also it is not clear that $[S, T; U]$ is embedded in $S *_U T$.

Indeed, it has long been known that some semigroup amalgams cannot be embedded in any semigroup. The first example, given by Kimura [24], is as follows. Let $U = \{u, v, 0\}$ be a three-element null semigroup (meaning that all products equal 0). Extend the multiplication of U to one of $S = U \cup \{s\}$ by defining $su = us = v$ and setting all other products equal to 0. Similarly, extend the multiplication of U to one of $T = U \cup \{t\}$ by defining $tv = vt = u$, and setting all other products equal to 0. The amalgam $[S, T; U]$ cannot be found in any semigroup. For suppose that W were a semigroup containing both S and T. Then, in W, we have $u = vt = sut = s0 = 0$.

The importance of the amalgamated free product lies in the fact that if $[S, T; U]$ can be embedded in any semigroup, then it can be embedded in $S *_U T$. Indeed, any pair of distinct members of $[S, T; U]$ which are identified in the amalgamated free product will also be identified in any attempt to embed $[S, T; U]$ in another semigroup. The amalgamated free product $S *_U T$ is the freest semigroup one can construct which respects all the relations implicit in the amalgam $[S, T; U]$. Therefore, $S *_U T$ is the test object used to study the nature of any collapse which must occur in the amalgam $[S, T; U]$ when it is embedded in another semigroup.

The amalgamated free product for groups, which has a similar construction, has found applications in the theory of group presentations by generators and relations. According to Magnus et al. this "eminently applicable group-theoretical construction... arises in a natural manner from a topological construction. If S_1 and S_2 are two arcwise connected spaces with fundamental groups F_1, F_2, then a space S arising from S_1, S_2 by identifying appropriate non-empty homeomorphic subspaces S_1' and S_2' of S_1 and S_2, respectively, may have as a fundamental group the free product of F_1 and F_2

with an amalgamated subgroup which is the fundamental group of both S_1' and S_2'".

Semigroup amalgams and their embeddability properties have come to the forefront in algebraic semigroup theory largely through the work of Howie and, more lately, Hall. Hall, in particular, predicts a large role will be played by the amalgamation property in the theory of presentations of inverse semigroups by generators and relations.

Although not every semigroup amalgam can be embedded in a semigroup without collapse, there are some powerful positive results which cannot be passed over. Those concerning epimorphisms and dominions will be considered in some detail later, but the result of Hall, that every amalgam of inverse semigroups can be embedded in an inverse semigroup without collapse, is exceptional and calls for attention, as in particular it includes Schreier's Theorem for group amalgams as a special case.

A member a of a semigroup S is called *regular* if there exists an inverse x for a in the Von Neumann sense that $a = axa$ and $x = xax$. A semigroup is called *regular* if all its members are regular. The prototype of a regular semigroup is $\mathscr{T}_X$, the semigroup of all functions on a set X to itself under composition. This so-called *full transformation semigroup* plays a role in semigroup theory akin to that of the symmetric group in group theory: there is a "Cayley Theorem" in that any semigroup S can be realized as a subsemigroup of $\mathscr{T}_{S^1}$ (where S^1 is the semigroup S with identity 1 adjoined if necessary). The embedding used mimics that employed in the Cayley Theorem: i: $s \to \varrho_s$, where ϱ_s: $S^1 \to S^1$ is defined by $x\varrho_s = xs$ $(x \in S^1)$. The presence of the identity in S^1 ensures that this embedding, called the *right regular representation* of S, is faithful.

A regular semigroup S is an *inverse semigroup* if each member $a \in S$ has a unique inverse a^{-1}. It is not very difficult to prove that a regular semigroup S is inverse if and only if the idempotents of S commute with each other (whereupon if we define $e \wedge f = ef$ for any two idempotents of S, the idempotents from a semilattice). Incidentally, the condition that the idempotents of S are central is the characteristic property of semilattices of groups (see [3] or [16]). The prototype of an inverse semigroup is the so-called *symmetric inverse semigroup* $\mathscr{I}_X$ on a set X, which consists of all one-to-one mappings whose domains and ranges are subsets of X (including the empty map). The product of two members of $\mathscr{I}_X$ is the partial composition map which results from composing them wherever possible. The "Cayley Theorem" for inverse semigroups goes under the title of the Preston–Wagner Theorem: any inverse semigroup can be embedded in $\mathscr{I}_X$ for some set X. The proof is non-trivial (see [3]).

That inverse semigroups are a major area of modern semigroup theory is attested to by the recent appearance of the monumental *Inverse Semigroups* by Petrich [27]. This book contains almost all work done on the subject up to the time of writing, and naturally includes Hall's Theorem on inverse semigroup amalgams. Schreier's Theorem is a corollary: given a group amalgam, Hall's Theorem allows it to be embedded in an inverse semigroup, S say. It is then easy to verify that the subsemigroup of S generated by the members of the amalgam is in fact a group, giving Schreier's result. In general, the embedding of a finite inverse semigroup amalgam into its free inverse product with amalgamation does not preserve finiteness (a counterexample, due to Ash, can be found in [16]). However, recently Hall has characterized those finite inverse semigroups U which are "strong amalgamation bases" in the category of all finite inverse semigroups (meaning that any amalgam of finite inverse semigroups with core U is embeddable in a finite inverse semigroup without collapse). From this result Schreier's Theorem for finite group amalgams is immediate. Hall's Theorem also tells us that inverse semigroups are absolutely closed in the category of all inverse semigroups, although to justify this we must return to the relationship between epis and amalgams.

Since semigroup amalgams can collapse when embedded in their free product, it becomes important to distinguish the ways in which this collapse can occur. We say that $[S, T; U]$ is *strongly embeddable* if it can be embedded in its free product without collapse. We say that $[S, T; U]$ is *weakly embeddable* if no two distinct members of S (and similarly of T) are identified with each other in the free product $S *_U T$. Note that weak embeddability does not preclude the possibility that a member s of $S \setminus U$ is identified with a member t of $T \setminus U$ in W (see Fig. 1).

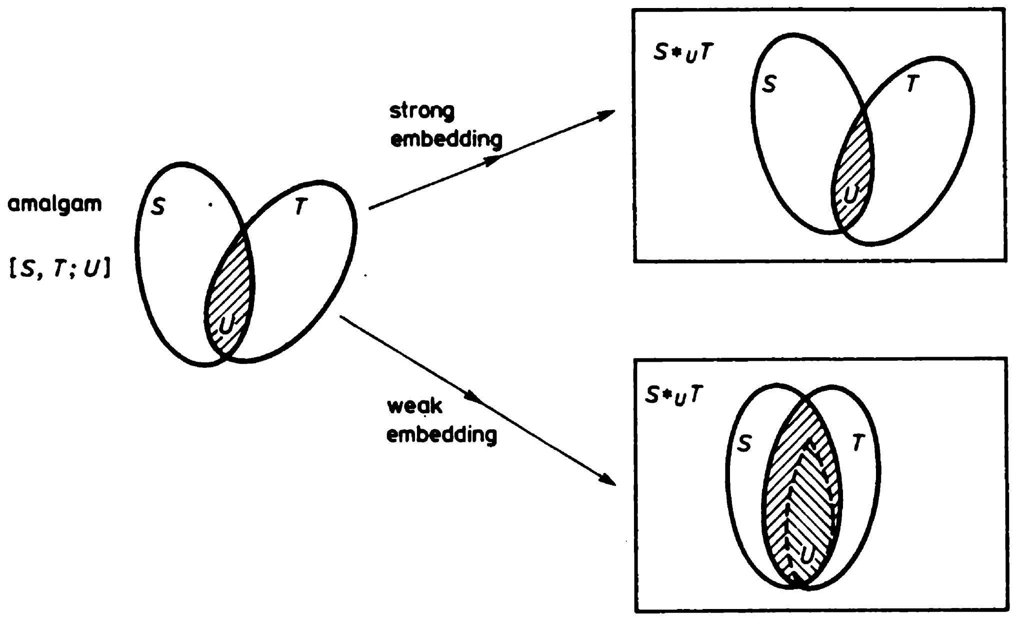

Fig. 1

Note that the Kimura example is not even weakly embeddable.

We say that a class of semigroups $\mathscr{C}$ has the *strong* [*weak*] *amalgamation property* if every amalgam of members of $\mathscr{C}$ can be strongly [weakly] embedded in some member of $\mathscr{C}$.

By a *special amalgam* we mean an amalgam $[S, S'; U]$, where there is an isomorphism between S and S' over U (meaning that U is fixed pointwise by this mapping). We say that a class $\mathscr{C}$ has the *special amalgamation property* if every special amalgam in the class can be strongly embedded in another member of $\mathscr{C}$. The strong amalgamation property is equivalent to the conjunction of the weak and special amalgamation properties. Let $[S, T; U]$ be an amalgam. By the weak amalgamation property there is a semigroup P such that $S, T \subseteq P$ and $U \subseteq S \cap T$. Consider a special amalgam $[P, P'; U]$ and invoke the special amalgam property to obtain a semigroup V such that $P, P' \subseteq V$ and $P \cap P' = U$. But copies of S and T are then embedded in V in such a way that $S \cap T = U$, as required. (This argument is universal algebraic in nature and a survey along these lines is to be found in [20].)

Before proceeding to the connection between amalgams and dominions we clarify the concept of an amalgamation base. Since the core of an amalgam is the unifying object in the structure, it is natural to inquire if there are any implications concerning embeddability which occur due to the character of the core of an amalgam. A semigroup U is a *weak* [*strong*] *amalgamation base* if every amalgam with core U is weakly [strongly] embeddable in a semigroup. Howie in [15] proved that every inverse semigroup is a strong amalgamation base. Another proof of this theorem can be found in [7] along with the remarkable fact that every weak amalgamation base is a strong amalgamation base. The proof is based on the representation extension property: U has the *representation extension property in S* if for any set X and any homomorphism $\varrho\colon U \to \mathscr{T}_X$ there exist a set Y, disjoint from X, and a homomorphism $\alpha\colon S \to \mathscr{T}_{X \cup Y}$ such that $\alpha_u | X = \varrho_u$ for all $u \in U$. We then say that U has the *representation extension property* if U has this property in every containing semigroup S. Hall proved that if U is a weak amalgamation base, then U has the representation extension property, which in turn implies that U is absolutely closed. Now, as we shall show, the absolute closedness of U is equivalent to U being a special amalgamation base (meaning that every special amalgam with core U is strongly embeddable), and, by the previous argument, the conjunction of the weak and special amalgamation base properties implies that U is a strong amalgamation base. We shall prove Hall's result at the end of the paper.

The connection between amalgams and dominions is via special amalgams. Let $[S, S'; U]$ be a special amalgam with an isomorphism $\alpha\colon S \to S'$ over U, and denote its free product with amalgamation by W. Obviously, $[S, S'; U]$ is weakly embeddable in W, as the amalgam can be weakly embedded in S. At this point it is worth pausing to re-examine the Kimura amalgam $[S, T; U]$ which is not weakly embeddable even though the semigroups S and T are isomorphic. There is no contradiction here as the isomorphism between S and T is not over U, since the elements u and v of U are interchanged under this mapping. The Kimura example is therefore not a special amalgam.

Returning to our special amalgam $[S, S'; U]$ we have $S, S' \subseteq W$ and $U \subseteq S \cap S'$. Take $d \in \mathrm{Dom}(U, S)$. Let β: $S \to S''$ be an isomorphism from S to a third copy S'' of S disjoint from W. Write U'' and d'' for $U\beta$ and $d\beta$, respectively. We may regard the homomorphisms

$$\beta^{-1}\colon S'' \to S \quad \text{and} \quad \beta^{-1}\alpha\colon S'' \to S'$$

as homomorphisms from S'' into W which agree on U''. It follows that

$$d = d''\beta^{-1} = (d''\beta^{-1})\alpha$$

(since $d'' \in \mathrm{Dom}(U'', S'')$), whence $d \in S \cap S'$ or, in other words, $\mathrm{Dom}(U, S) \subseteq S \cap S'$.

Significantly, the converse inclusion holds. Denote the amalgam $[S, S'; U]$ by A. Let α, β: $S \to T$ be two homomorphisms from S such that $\alpha|U = \beta|U$, and suppose that $d\alpha \neq d\beta$ $(d \in S)$. The amalgam

$$B = [S\alpha, S\beta; S\alpha \cap S\beta]$$

is embedded in T, and so we may without loss take T to be the free product with amalgamation of B. There is then a surmorphism γ: $S *_U S' \to T$ in which $s\gamma = s\alpha$, $s'\gamma = s'\beta$ $(s \in S, s' \in S')$. The intuitive justification for this is the fact that $S *_U S'$ and T are the freest semigroups generated by the amalgams A and B, respectively, which respect all the relations defined by the amalgams, and it is plain that the amalgam B induces all the relations of A (and perhaps more) (for details see [16], Chapter VII). Since $d\alpha \neq d\beta$, we have $d \notin S \cap S'$ in $S *_U S'$, and we conclude that $\mathrm{Dom}(U, S)$ is exactly the subsemigroup of S consisting of all members which are identified with their counterparts in S' in $S *_U S'$.

Our epimorphism-related concepts can now be reworded in the language of semigroup amalgams: U is absolutely closed means that U is a special amalgamation base, U is dense in S means that $S *_U S'$ is isomorphic to S. What is more, theorems about amalgams can have corollaries for epis: every inverse semigroup is an amalgamation base implies that inverse semigroups are absolutely closed, every amalgam of inverse semigroups is strongly embeddable in an inverse semigroup implies that inverse semigroups are absolutely closed in the category of all inverse semigroups (neither of these results is a corollary to the other).

However, this characterization of the dominion is equally valid in any category of algebras where free products with amalgamation exist and share the same universal properties. Hence it is asking too much to expect this theorem to enable us to recognize any epimorphism or, more generally, how to decide whether or not a given $d \in \mathrm{Dom}(U, S)$. This perhaps tells us that d will be dominated by U if d can be factorized in some special way in which the elements of U play a central role. This is still far from the celebrated Zigzag Theorem of Isbell.

ZIGZAG THEOREM FOR SEMIGROUPS. *Let U be a subsemigroup of a semigroup S. Then $d \in \mathrm{Dom}(U, S)$ if and only if $d \in U$ or there is a series of factorizations of d:*

$$d = x_1 u_0 = x_1 u_1 y_1 = x_2 u_2 y_1 = \ldots = x_m u_{2m-1} y_m = u_{2m} y_m$$

with

$$u_0 = u_1 y_1, \quad x_i u_{2i-1} = x_{i+1} y_{2i}, \quad u_{2i} y_i = u_{2i+1} y_{i+1} \quad (1 \leqslant i \leqslant m-1),$$

$$x_m u_{2m-1} = u_{2m} \quad (u_i \in U,\ x_i,\ y_i \in S).$$

Such a sequence of factorizations is called a *zigzag* in S over U with *value d*, *length m* and *spine* the list $u_0, u_1, \ldots, u_{2m}$.

For example, consider the epimorphic embedding i: $(0, 1] \to (0, \infty)$ mentioned above. A zigzag for $x > 1$ is given by

$$\begin{aligned} x = x \cdot 1 \quad &= x u_0, \\ x \cdot x^{-1} \cdot x &= x u_1 y, \\ \hat{1} \cdot x \quad &= \hat{u}_2 y, \end{aligned}$$

where the braces denote zigzag equalities. Length-one zigzags suffice here, but this is not always so (see, e.g., [13]).

In one direction the proof of Isbell's Theorem is easy: given a zigzag Z for d it is routine to verify that if α, β: $S \to T$ are homomorphisms which agree on U, then $d\alpha = d\beta$. The argument is just a generalization of the manipulations used in our example at the close of the first section. The converse is obviously difficult, for how is one to extract a zigzag from the condition that $d \in \mathrm{Dom}(U, S) \setminus U$? The first proof by Isbell [18] was topological in nature and was incomplete. The proof was made rigorous by Philip [28]. An algebraic proof was given by Storrer [31] and is based on work by Stenstrom [30]. Howie's book [16] contains a proof along these lines. It is similar in idea to proofs of Zigzag Theorems for Rings and Unital Rings which also rely on tensor product ideas [29]. Recently, a different proof of Isbell's Theorem has been shown to the author by D. Jackson. This proof is remarkable in that it uses the method of "regular diagrams" which is "closely related to the geometric methods of Lyndon and Schupp in combinatorial group theory". Jackson also uses an HNN construction, first developed in a semigroup context by Howie [14]. The HNN construction seems to fill the role played by the tensor product in Storrer's proof. Both proofs are similar in that they proceed to the Zigzag Theorem for Semigroups via the corresponding theorem for Monoids (semigroups with identity).

One might hope to prove the Zigzag Theorem by a grappling directly with $S *_U S'$, but no one has as yet succeeded. However, this can be done in the category of Commutative Semigroups. First, note that the Zigzag Theorem for Commutative Semigroups is not a corollary to the Zigzag Theorem for Semigroups: it is conceivable that there exists $d \in S \setminus U$ which is dominated by U with respect to all pairs of morphisms from S into a *commutative* semigroup T, but there exist α, β: $S \to V$, V not commutative, such that $\alpha|U = \beta|U$ but $d\alpha \neq d\beta$. Indeed, whether or not zigzags fully describe dominions in the category of Bands (semigroups of idempotents) is still an open question, made more interesting by the discovery by Hall [8] of infinitely many varieties of semigroups in which the Zigzag Theorem does not hold. **(P 1350)**

The proof of the Zigzag Theorem for the category of Commutative Semigroups, taken from [17], begins by noting that the free product $S * T$ of two commutative semigroups can be regarded as the direct product $S^{(1)} \times T^{(1)}$, with the identity (1, 1) removed ($S^{(1)}$ means S with an adjoined identity, whether or not it already has one). Now, if $d \in \mathrm{Dom}(U, S)$, there is a sequence of transitions $(1, d) \to \ldots \to (d, 1)$, where a typical transition $(x, y) \to (z, t)$ is either

$$\text{an } r\text{-step, } (x, y) = (p, q)(u, 1)(r, s),\ (z, t) = (p, q)(1, u)(r, s),$$

or

$$\text{an } l\text{-step, } (x, y) = (p, q)(1, u)(r, s),\ (z, t) = (p, q)(u, 1)(r, s),$$

where $u \in U$, $p, q, r, s \in S^{(1)}$. By commutativity we have $x = zu$ and $uy = t$ in case of an r-step; and $z = ux$ and $y = ut$ if it is an l-step. Obviously, two successive r-steps (corresponding to u and u', respectively) can be abbreviated to a single r-step (corresponding to $u'u$); a dual remark applies to l-steps. Hence we may assume that r- and l-steps occur alternately in the sequence. Since the adjoined identity 1 has no divisors in U, the first and last members of the sequence are l-steps. Therefore, there is an odd number (say $2m+1$) of steps, the corresponding factorizations being necessarily of the form

$$\begin{aligned}
d &= x_1 u_0, & u_0 &= u_1 y_1,\\
x_1 u_1 &= x_2 u_2, & u_2 y_1 &= u_3 y_2,\\
&\cdots\cdots\cdots & &\cdots\cdots\cdots\\
x_{m-1} u_{2m-3} &= x_m u_{2m-2}, & u_{2m-2} y_{m-1} &= u_{2m-1} y_m,\\
x_m u_{2m-1} &= u_{2m}, & u_{2m} y_m &= d
\end{aligned}$$

with each $u_i \in U$. This completes the proof of the Zigzag Theorem for Commutative Semigroups.

It is noteworthy that neither Storrer's nor Jackson's proof of the Zigzag Theorem for Semigroups seems to be adaptable to the category of Commutative Semigroups.

In passing we state the Zigzag Theorem for the category of Rings. Let R be a subring of a ring S. Then $d \in \mathrm{Dom}(R, S)$ if and only if $d = a + XPY$, where $a \in R$, X is a row vector over S, Y a column vector over S, P a matrix over R^1 such that both XP and PY are matrices over R. For the category of Rings with Unity (homomorphisms must preserve the identity) a similar Zigzag Theorem holds but there is the simplification that the element a can be taken to be zero (see [29]).

3. Some applications. The efficacy of zigzag theorems is demonstrated through four examples.

Since epimorphisms are akin to surmorphisms, they might be expected to have structure-preserving qualities. Bulaszewska and Krempa [1] proved that the epimorphic image of a commutative ring is commutative. The corresponding theorem for Semigroups is an easy consequence of the Zigzag Theorem.

THEOREM (Isbell [18]). *Let α: $U \to S$ be an epimorphism from a commutative semigroup U to a semigroup S. Then S is commutative.*

Proof. Consider the epimorphic inclusion i: $U\alpha \to S$. Let $u \in U\alpha$, $d \in S \setminus U\alpha$, and let Z be a zigzag in S over $U\alpha$ with value d, as in the Zigzag Theorem. Then $U\alpha$ is commutative and

$$du = x_1 u_0 u = x_1 u u_0 = x_1 u u_1 y_1 = x_1 u_1 u y_1 = x_2 u_2 u y_1 = \ldots$$
$$= x_m u u_{2m-1} y_m = x_m u_{2m-1} u y_m = u_{2m} u y_m = u u_{2m} y_m = ud.$$

Hence $U\alpha$ is central in S. It remains to show that any two members $d, t \in S \setminus U$ commute. However, since t commutes with all members of the spine of the zigzag for d, we can repeat the above argument to prove that $dt = td$, as required.

The above proof in fact shows that the dominion of a commutative semigroup is itself commutative.

N. M. Khan has generalized the preceding theorem in two directions by showing that any equational class of commutative semigroups is closed under the taking of epis [22] and that every permutation identity is respected by epimorphisms [23].

A semigroup S is *right* [*left*] *simple* if $aS = S$ [$Sa = S$] for all $a \in S$. A global definition of a group, often utilized in semigroup theory, is that of a semigroup which is both left and right simple. One might therefore expect right simple semigroups to resemble groups. The classical result along these lines is that the following are equivalent:

(i) S is right simple and left cancellative;

(ii) S is right simple with at least one idempotent;

(iii) S is isomorphic to $G \times R$, where G is a group and R is a right zero semigroup (meaning that $xy = y$ for all $x, y \in R$).

This characterization of these semigroups, known as *right groups*, is given in Clifford and Preston Vol. I [3]. Right simple idempotent-free semigroups exist, and the standard example is the Baer–Levi semigroup, which is the subsemigroup of $\mathscr{T}_X$, with X infinite and countable, consisting of all injections whose ranges have infinite complements in X. It is elementary to check that the Baer–Levi semigroup is right simple, right cancellative with no idempotent.

However, all right simple semigroups enjoy the property of being special amalgam bases. The proof of this begins with the observation that two zigzags Z and Z' in S over U with the same spine are *equivalent* in the sense that they have the same value. Call their respective values d and d'. Let Z be of the form given in the Zigzag Theorem and Z' be defined by

$$d' = t_1 u_0 = t_1 u_1 z_1 = t_2 u_2 z_1 = \ldots = t_m u_{2m-1} z_m = u_{2m} z_m.$$

We obtain

$$d = x_1 u_0 = x_1 u_1 z_1 = x_2 u_2 z_1 = x_2 u_3 z_2 = \ldots = x_m u_{2m-1} z_m = u_{2m} z_m = d'.$$

THEOREM (Howie and Isbell [17]). *A right simple semigroup U is absolutely closed.*

Proof. Suppose that U is contained in a semigroup S. The result is proved by showing that a given zigzag Z in S over U is equivalent to a zigzag Z':

$$x_1 u_0 = x_1 u_1 v_1 = x_2 u_2 v_1 = x_2 u_2 v_1 = \ldots = u_{2m} v_m,$$

with $v_1, v_2, \ldots, v_m \in U$. The common value of both zigzags is then $u_{2m} v_m \in U$, whence U is closed in S.

Now, the right simplicity of U means exactly that there exists a solution in U to any equation of the form $ax = b$ $(a, b \in U)$. In particular, there exists $v_1 \in U$ such that $u_0 = u_1 v_1$, and then we can find $v_2 \in U$ such that $u_2 v_1 = u_3 v_2$, and so on, producing the required list of elements $v_1, v_2, \ldots, v_m \in U$.

Next, we give a quick application of the Zigzag Theorem for Rings with Unity due to Gardner [5]. A ring R is *regular* if its multiplicative semigroup is regular. The class of regular rings includes all division rings and the ring $M_n(R)$ of all $(n \times n)$-matrices over a regular unital ring is itself regular [21].

THEOREM (Gardner [5]). *Let α: $R \to S$ be an epimorphism from a regular unital ring R to a unital ring S in the category of Rings with Unity. Then $R = S$.*

Proof. Since the morphic image of a regular ring is clearly regular, and the inclusion $i\colon R\alpha \to S$ is a dense embedding, we can assume without loss that α is an embedding. Take $s \in S$, which by the Zigzag Theorem for Unital Rings has the form XPY, where XPY is a (ring) zigzag. By inserting rows or columns of zeros if necessary, we may assume that P is a square, $n \times n$ say, matrix. By the above comment, P is itself regular, and so $P = PTP$, where T is a matrix over R. But then

$$s = XPY = (XP)\,T(PY) \in R,$$

as XP and PY are matrices over R. Therefore $R = S$.

This neat proof belies the difficulty of extracting the corresponding result for the category of all Rings [5]. However, the same result for semigroups is false. Indeed, there exists a semigroup consisting entirely of idempotents which can be properly densely embedded in another semigroup [10]. However, any epimorphism from a finite regular semigroup is onto in the category of Semigroups (see [9] or [11]).

Our final application is the promised proof of Hall's Theorem [7].

THEOREM. *Any semigroup U which is a weak amalgamation base is a strong amalgamation base.*

This result contrasts with the category of Distributive Lattices where all objects are weak amalgamation bases, but not all have the strong amalgamation base property [6].

Proof. It suffices to prove that if U is not absolutely closed, then it is not a weak amalgamation base, because the strong amalgamation base property is the conjunction of the weak and special amalgamation base properties. To this end suppose that S contains U as a subsemigroup with Z a zigzag in S over U with value $d \in S \setminus U$. Let X be a special amalgam $[S', S''; U]$ considered as a set (S', S'' copies of S). Without loss we suppose that $S = S^1$. Define a map $\varrho\colon U \to \mathscr{T}_X$ whereby $u \to \varrho_u$, where $s'\varrho_u = s'u$, $s''\varrho_u = s''u$ ($s' \in S'$, $s'' \in S''$). Clearly, ϱ is an isomorphism into $\mathscr{T}_X$. Let T be the subsemigroup of $\mathscr{T}_X$ consisting of $U\varrho$ together with the constant maps, and consider the amalgamated free product $S *_U T$. Denote the two distinct members of X corresponding to d by d' and d''. Use single and double primes to denote members of S' and S'', respectively. In the following calculation we write x_1', x_2', etc. to represent the members of S', and $\bar{x}_1'$, $\bar{x}_2'$, etc. to represent the corresponding constant maps. Bearing in mind that u and ϱ_u are identified in $S *_U T$ we have

$$\begin{aligned}\bar{d}' &= \overline{(x_1' u_0)} = \bar{x}_1' \varrho_{u_0} = \bar{x}_1' u_0 = \bar{x}_1' u_1 y_1 = \bar{x}_1' \varrho_{u_1} y_1 = \overline{(x_1' u_1)}\, y_1 = \overline{(x_2' u_2)}\, y_1 \\ &= \bar{x}_2' \varrho_{u_2} y_1 = \bar{x}_2' u_2 y_1 = \bar{x}_2' u_3 y_2 = \ldots = \bar{x}_m' \varrho_{u_{2m-1}} y_m = \overline{(x_m' u_{2m-1})}\, y_m \\ &= \bar{u}_{2m} y_m.\end{aligned}$$

However, by symmetry, we also get $\bar{d}'' = \bar{u}_{2m} y_m$. In other words, $d' = d''$ in $S *_U T$, showing that the amalgam $[S, T; U]$ is not weakly embeddable, as required.

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