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SOME COMPACTIFICATIONS OF GENERAL ALGEBRAS

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In this paper I give several remarks connected with the theory of algebraically compact Abelian groups and the theory of $\mathfrak{m}$-universal and $\mathfrak{m}$-homogeneous or $\mathfrak{m}$-replete relational systems. No deep new results are given in this paper but some easy facts mentioned here are perhaps new, in particular a remark that an argument of Fuchs [5] can be much generalized (Theorem 1) and a connection of $\mathfrak{m}$-repleteness with a kind of compactness analogous to this of algebraically compact Abelian groups (Theorem 3).

1. Terminology and notation. For any sets X and Y, Y^X denotes the set of all functions $f: X \to Y$. A theory of ordinal numbers will be supposed in which $a = \{\xi : \xi < a\}$, e. g. $n = \{0, \ldots, n-1\}$ for $n < \omega$.

$\mathfrak{A} = \langle A, \{F_t\}_{t\in T}\rangle$ is called an *algebra* (or *general algebra*) if A is a non-empty set and $F_t : A^{f(t)} \to A$, where $f(t) < \omega$ for every $t \in T$ (if $f(t) = 0$ we mean that $F_t \in A$). The pair $\langle T, f\rangle$ is called the *similarity type* of $\mathfrak{A}$. We write often $a \in \mathfrak{A}$ for $a \in A$.

The set of variables (*unknowns*) will be any indexed system x_s $(s \in S)$, where S is any set, not necessarily denumerable. For any terms τ and ϑ constructed with these variables and function symbols corresponding to our similarity type the equation $\tau = \vartheta$ defines a relation $R \subseteq A^S$, which is the set of all systems a_s $(s \in S)$ which satisfy it. By an *equation with constants in* $\mathfrak{A}$ we mean an equation $\tau = \vartheta$ in which some variables are replaced by constant elements of A. This also defines a relation $R \subseteq A^S$ (which does not depend on those variables which were replaced by constants). We shall use the same letter R to denote an equation or the corresponding relation. A system of equations, finite or infinite with or without constants, is called *solvable* if the intersection of the corresponding relations is non-empty, and a system belonging to this intersection is called a *solution*.

If $\mathfrak{A}_u$ $(u \in U)$ is a system of algebras of the same similarity type (*similar*) and I is an ideal of subsets of U, then $\mathfrak{P}_{u\in U} \mathfrak{A}_u/I$ denotes the *reduced direct product*, i. e. the quotient of the direct product $\mathfrak{P}_{u\in U} \mathfrak{A}_u$ by a congruence $\equiv$ defined as follows:

$$a \equiv b \leftrightarrow \{u : a(u) \neq b(u)\} \in \boldsymbol{I}.$$

An algebra $\mathfrak{A}$ is called *equationally* $\mathfrak{m}$*-compact* (*weakly equationally* $\mathfrak{m}$*-compact*), or shortly $\mathfrak{m}$*-compact* (*weakly* $\mathfrak{m}$*-compact*), if each system of $\mathfrak{m}$ equations with constants in $\mathfrak{A}$ (without constants), with any set S, is solvable whenever any finite subsystem of this system is solvable. $\mathfrak{A}$ is called *compact* (*weakly*) if it is $\mathfrak{m}$-compact (weakly) for any cardinal $\mathfrak{m}$.

$\mathfrak{R} = \langle A, \{R_t\}_{t\in T}\rangle$ is called a *relational system* if A is a non-empty set and $R_t \subseteq A^{r(t)}$, where $r(t)$ is a positive integer for every $t \in T$. The pair $\langle T, r\rangle$ is called the *similarity type* of $\mathfrak{R}$. Sometimes $\overline{\overline{\mathfrak{R}}}$ will denote $\overline{\overline{A}}$. We denote by $L_S^{\langle T,r\rangle}$ the set of all formulas of the corresponding first order predicate calculus with equality with a system of variables x_s $(s\in S)$ (therefore $L_\omega^{\langle T,r\rangle}$ is the classical calculus). For the notion of satisfaction of a formula of $L_S^{\langle T,r\rangle}$ by a system a_s $(s\in S)$ of elements of A see [23]; for the notions of elementary equivalence and elementary extensions of relational systems see [23]; for the definition of reduced products of similar relational systems see [4], an extensive theory of this operation is developed in [4], [10], [11], [13], [15]. By a *formula with constants* in $\mathfrak{R}$ we mean any formula $\varphi \in L_S^{\langle T,r\rangle}$ in which some free variables are replaced by constant elements of A. The satisfaction of a formula with constants by a system a_s $(s\in S)$ is defined in the usual way, i.e. the remaining free variables x_s are replaced by the corresponding a_s, etc.

A relational system $\mathfrak{R}$ is called *elementarily* $\mathfrak{m}$*-compact* (*weakly elementarily* $\mathfrak{m}$*-compact*) if each set of $\mathfrak{m}$ formulas with constants in $\mathfrak{R}$ (without constants), with any set S, is satisfiable by a system a_s $(s\in S)$ of elements of $\mathfrak{R}$ whenever all finite subsets of this set are satisfiable by such systems.

2. Introduction. The above notions of compactness were studied hitherto only for such Abelian groups, for which $\aleph_0$-compactness implies compactness (Łoś [16]). The theory of these Abelian groups was developed in [1], [2], [5], [9] and [16] (these groups are called there algebraically compact, consistently with an earlier terminology of I. Kaplansky), e. g. every divisible Abelian group is such a group. Note that the infinite cyclic group is not $\aleph_0$-compact as it is shown by the following system of equations:

$$3x_0 + x_1 = 1, \quad x_1 = 2x_2, \quad x_2 = 2x_3, \quad \ldots$$

It was also proved by Łoś [17] that for every regular cardinal number $\mathfrak{m}$ of 01-measure 0 there is a system of $2^{\mathfrak{m}}$ linear equations with integral coefficients and constants such that every subsystem of it having less than $\mathfrak{m}$ equations can be solved but the whole system can not. The ring of integers has a still stranger property: there exists a system of $\aleph_1$ equations with constants every denumerable subsystem of which is solvable but the whole system is not. Such is the system

$$x_{\xi,\eta}(5z_\xi+2)+y_{\xi,\eta}(5z_\eta+2) = 1 \qquad (\xi, \eta < \omega_1,\ \xi \neq \eta),$$

where $x_{\xi,\eta}$, $y_{\xi,\eta}$ and z_ζ are the unknowns. In fact, such an equation clearly implies $z_\xi \neq z_\eta$ and hence the whole system is not solvable; but a denumerable subsystem involves only denumerably many z's, e. g. $z_{\zeta_1}, z_{\zeta_2}, \ldots$, then we take for z_{ζ_i} such integers that the numbers $5z_{\zeta_i}+2$ are different primes (thus using the theorem of Dirichlet) and then obviously adequate x's and y's for the subsystem can be also found (1). It is not known if the infinite cyclic group has this property, i. e. if such an uncountable system of linear equations can be produced (**P 482**). Note that no infinite field $\mathfrak{F}$ is a compact ring. In fact, the system

$$(x-a)y_a = 1, \qquad a \in \mathfrak{F},$$

of ring-equations with constants in $\mathfrak{F}$, where the unknowns are x and all y_a, has no solution, however, each finite subsystem of it clearly has. Finally, let us mention the possibility of infinite systems of equations such that no infinite subsystem of which is solvable, but all finite subsystems are. Such is the system

$$x = y_n^2+n, \qquad n = 1, 2, \ldots,$$

of equations with constants in the ring of real numbers.

Given an equational class $\boldsymbol{K}$ of similar algebras (see e. g. [21] for a definition of equational classes) and an algebra $\mathfrak{A} \in \boldsymbol{K}$, the problem arises if there exists a *compactification* $\mathfrak{B} \in \boldsymbol{K}$ of $\mathfrak{A}$, i. e. an algebra $\mathfrak{B} \in \boldsymbol{K}$ which is compact and has a subalgebra isomorphic to $\mathfrak{A}$.

The answer in general is negative even for weak compactification. In fact, let $\mathfrak{A} = \langle\{0, 1, 2, \ldots\}, \{F_0, F_1, F_2\}\rangle$, where

$$F_0 = 0, \qquad F_1 = 1, \qquad F_2(x, y) = \begin{cases} 0 \text{ for } x = y, \\ 1 \text{ for } x \neq y \end{cases}$$

and let $\boldsymbol{K}$ be the equational class with the similarity type $\langle\{0, 1, 2\}, f\rangle$, where $f(0) = f(1) = 0$, $f(2) = 2$ and defined by a single equation

$$F_2(x, x) = F_0.$$

(1) The above property of the ring of integers was discovered several years ago by A. Ehrenfeucht, but his system of equations is not known to the author. Related set theoretical problems are treated by Erdős and Hajnal [3] (section 8).

Clearly $\mathfrak{A} \in \boldsymbol{K}$ but $\mathfrak{A}$ has no compactification in $\boldsymbol{K}$, since every finite subsystem of a system of equations

$$F_2(x_{s_1}, x_{s_2}) = F_1 \qquad (s_1, s_2 \in S,\ s_1 \neq s_2)$$

is solvable in $\mathfrak{A}$ but an algebra $\mathfrak{B} \in \boldsymbol{K}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and in which this system is solvable must have at least the cardinality of S.

On the other hand, we will see (Theorem 5) that for every cardinal $\mathfrak{m}$ every algebra and, moreover, every relational system has an elementarily $\mathfrak{m}$-compact elementary extension.

A *topological algebra* is an algebra $\langle A, \{F_t\}_{t \in T}\rangle$ enriched by a Hausdorff topology in the set A, such that all F_t are continuous; it is called *compact topological* if this topology is compact. The following proposition visibly follows from the Tihonov product theorem:

PROPOSITION 1. *A compact topological algebra is a compact algebra.*

This permits often to get a compactification of an algebra by constructing a topological compactification. It applies to all the finitely approximable algebras, i. e. algebras $\mathfrak{A}$ such that for every pair a, b of its distinct elements there is a homomorphism h of $\mathfrak{A}$ onto a finite algebra with $h(a) \neq h(b)$. Indeed, $\mathfrak{A}$ is then imbeddable in the direct product of all these finite algebras $h(\mathfrak{A})$ which, by the theorem of Tihonov, has a compact topology. Many algebras are known to be finitely approximable. It follows from the theorem of Birkhoff-Tarski [21] that such are all free algebras in an equational class generated by a set of finite algebras, e. g. free groups, solvable free groups, some free Abelian rings etc. See also [6], [8], [18] and [19] for several other important classes of algebras which are finitely approximable. Topological compactification can be also applied to every Abelian group. This method fails for groups in general, since there are groups which are not imbeddable into any compact topological group. Such is, as shown by H. Freudenthal (see [7]), e. g. the group of linear substitutions $ax+b$ with a, b rational and $a \neq 0$, for which the problem of existence of a compactification in the class of all groups is open (**P 483**).

Nothing is known about compactness or $\mathfrak{m}$-compactness of non-Abelian connected locally compact topological groups.

Let us still add the following easy general statements:

PROPOSITION 2. *The direct product of $\mathfrak{m}$-compact (weakly $\mathfrak{m}$-compact) similar algebras is an $\mathfrak{m}$-compact (weakly $\mathfrak{m}$-compact) algebra.*

PROPOSITION 3. *A retract of an $\mathfrak{m}$-compact algebra, i. e. its image by an endomorphism which is an identity on this image, is $\mathfrak{m}$-compact. If h is an endomorphism of an algebra $\mathfrak{A}$, then $\mathfrak{A}$ is weakly $\mathfrak{m}$-compact if and only if $h(\mathfrak{A})$ is.*

In connection with this proposition the following unsolved problem should be mentioned: Is every compact algebra a retract of a compact topological algebra? (**P 484**). For Abelian groups the answer is affirmative as shown by Łoś [16].

3. Theorems on equational compactness.

THEOREM 1. *If* $\mathfrak{A}_u$ $(u \in U)$ *is a system of similar algebras, and* $\boldsymbol{I}$ *is an* $\aleph_a$*-additive* (²) *ideal of subsets of* U *such that* U *is a union of* $\aleph_a$ *members of* $\boldsymbol{I}$, *then the reduced direct product* $\mathfrak{P}_{u \in U} \mathfrak{A}_u/\boldsymbol{I}$ *has the following property: each system of* $\aleph_a$ *equations with constants in* $\mathfrak{P}_{u \in U} \mathfrak{A}_u/\boldsymbol{I}$ *such that all its subsystems having less than* $\aleph_a$ *equations are solvable is solvable.*

Proof (³). Let R_ξ $(\xi < \omega_a)$ be a system of equations with constants in $\mathfrak{P}_{u \in U} \mathfrak{A}_u/\boldsymbol{I}$ with a system $\{x_s\}_{s \in S}$ of unknowns and let C be the set of all the constants appearing in this system. For every $\boldsymbol{c} \in C$ we choose a representant $c \in \boldsymbol{c}$ which is an element of the algebra $\mathfrak{P}_{u \in U} \mathfrak{A}_u$, and $c(u)$ denotes the value of c on the axis $\mathfrak{A}_u$. Let R_ξ^u denote an equation with constants in $\mathfrak{A}_u$ obtained from R_ξ by replacing each $\boldsymbol{c}$ occurring in R_ξ by the corresponding $c(u)$. By the supposition of the theorem for every $\eta < \omega_a$ there exists a solution $\{\boldsymbol{a}_s^\eta\}_{s \in S}$ of the system R_ξ $(\xi < \eta)$. Let $a_s^\eta \in \boldsymbol{a}_s^\eta$ for any $\eta < \omega_a$, $s \in S$.

The following known property of equations in reduced products holds (see [3]):

$$\bigwedge_{s \in S} (a_s \in \boldsymbol{a}_s) \to \big(\{\boldsymbol{a}_s\}_{s \in S} \in R_\xi \leftrightarrow \{u : \{a_s(u)\}_{s \in S} \notin R_\xi^u\} \in \boldsymbol{I}\big). \tag{1}$$

Therefore, putting $B_\zeta^\eta = \{u : \{a_s^\eta(u)\}_{s \in S} \notin R_\zeta^u\}$, we get

$$B_\zeta^\eta \in \boldsymbol{I} \quad \text{for any } \zeta < \eta. \tag{2}$$

By the assumption concerning $\boldsymbol{I}$ it is easy to produce a sequence $A_\xi \in \boldsymbol{I}$ $(\xi < \omega_a)$ such that

$$A_{\xi_1} \cap A_{\xi_2} = 0 \text{ for } \xi_1 \neq \xi_2, \quad \bigcup_{\xi < \omega_a} A_\xi = U \quad \text{and} \quad B_\zeta^\eta \subseteq \bigcup_{\xi < \eta} A_\xi \text{ for } \zeta < \eta. \tag{3}$$

Given such a sequence, let us put

$$a_s(u) = a_s^{\xi(u)}(u), \tag{4}$$

where $\xi(u)$ is defined by $u \in A_{\xi(u)}$.

(²) i. e. every union of less than $\aleph_a$ members of $\boldsymbol{I}$ is a member of $\boldsymbol{I}$.

(³) This proof is an adaptation of an argument of Fuchs [5].

By (1) the theorem will be proved if we show for each $\eta < \omega_\alpha$ that

$$M_\eta = \{u : \{a_s(u)\}_{s\in S} \notin R_\eta^u\} \in \boldsymbol{I},$$

and we will show this by proving that $M_\eta \subseteq \bigcup_{\xi<\eta} A_\xi$. Indeed, by (4)

$$M_\eta = \{u : \{a_s^{\xi(u)}(u)\}_{s\in S} \notin R_\eta^u\}$$

and if $u \notin \bigcup_{\xi<\eta} A_\xi$, then $\xi(u) \geqslant \eta$ and by (2) and (3) (the inclusion) we have $\{a_s^{\xi(u)}(u)\}_{s\in S} \in R_\eta^u$, i. e. $u \notin M_\eta$, q. e. d.

Remark. This theorem may be still generalized if each algebra $\mathfrak{A}_u$ has a one-element subalgebra $\mathfrak{E}_u$. Then, given any ideal $\boldsymbol{J}$ of subsets of U, we denote by $\mathfrak{P}^{(\boldsymbol{J})}_{u\in U}\mathfrak{A}_u$ the subalgebra of $\mathfrak{P}_{n\in U}\mathfrak{A}_u$ consisting of all those elements a for which

$$\{u : a(u) \notin \mathfrak{E}_u\} \in \boldsymbol{J}.$$

Now the conclusion of our theorem is valid for any algebra of the form $\mathfrak{P}^{(\boldsymbol{J})}_{u\in U}\mathfrak{A}_u/\boldsymbol{I}$, where $\boldsymbol{J}$ is the $\aleph_{\alpha+1}$-additive ideal generated by $\boldsymbol{I}$ ($\boldsymbol{I}$ is $\aleph_\alpha$-additive). This generalization in the case of $\alpha = 0$ and $\mathfrak{A}_u$ being Abelian groups is the theorem of Fuchs [5].

THEOREM 2. *Suppose that the assumptions of Theorem* 1 *on* $\mathfrak{A}_u$ $(u \in U)$, U *and* $\boldsymbol{I}$ *are satisfied and, moreover, that*

$$\{u : \mathfrak{A}_u \text{ is not } \aleph_\beta\text{-compact (weakly } \aleph_\beta\text{-compact)}\} \in \boldsymbol{I} \textit{ for every } \beta < \alpha.$$

Then $\mathfrak{P}_{u\in U}\mathfrak{A}_u/\boldsymbol{I}$ *is* $\aleph_\alpha$*-compact (weakly* $\aleph_\alpha$*-compact).*

Proof. By Theorem 1 it is enough to show that if R_ξ $(\xi < \omega_\beta)$, where $\beta < \alpha$, is a system of equations with constants in $\mathfrak{P}_{u\in U}\mathfrak{A}_u/\boldsymbol{I}$ (without constants) every finite subsystem of which is solvable, then this system is solvable. Let R_ξ^u denote an equation with (without) constants in $\mathfrak{A}_u$ defined by means of R_ξ in the same way as in the proof of Theorem 1. Let

$$B = \{u : \text{there exists a finite sequence } \xi_1, \ldots, \xi_n < \omega_\beta \text{ such that the system } R_{\xi_1}^u, \ldots, R_{\xi_n}^u \text{ is not solvable}\}.$$

It would be enough to show that $B \in \boldsymbol{I}$, since then the system R_ξ^u $(\xi < \omega_\beta)$ is solvable whenever

$$u \notin B \cup \{u : \mathfrak{A}_u \text{ is not } \aleph_\beta\text{-compact (weakly } \aleph_\beta\text{-compact)}\},$$

which shows that the system R_ξ $(\xi < \omega_\beta)$ is solvable.

To prove $B \in \boldsymbol{I}$ let us remark that

$$B = \bigcup_{n=1}^{\infty} \bigcup_{\xi_1, \ldots, \xi_n < \omega_\beta} \{u : \text{the system } R^u_{\xi_1}, \ldots, R^u_{\xi_n} \text{ is not solvable}\}$$

and, since each system $R_{\xi_1}, \ldots, R_{\xi_n}$ is solvable,

$$\{u : \text{the system } R^u_{\xi_1}, \ldots, R^u_{\xi_n} \text{ is not solvable}\} \in \boldsymbol{I}.$$

Hence, since $\boldsymbol{I}$ is $\aleph_\alpha$-additive, we get $B \in \boldsymbol{I}$, q. e. d.

By a repeated application of the operation $\mathfrak{A}^{\omega_n}/\boldsymbol{I}$, where $\boldsymbol{I} = \{X : X \subset \omega_n, \overline{\overline{X}} < \aleph_n\}$, and by Theorems 1 and 2 we can prove for any $n < \omega$ that any algebra $\mathfrak{A}$ has an $\aleph_n$-compactification belonging to the equational class generated by $\mathfrak{A}$. But if the maximal ideals are used, then much better results can be obtained as we will see in the next section.

4. Theorems on elementary compactness. Let us recall a notion quite analogous to elementary $\mathfrak{m}$-compactness, which was introduced by Keisler in [12]: a relational system $\mathfrak{R}$ is called $\mathfrak{n}$-*replete* if each system of less than $\mathfrak{n}$ formulas with constants in $\mathfrak{R}$ each of them involving only one and the same free variable is satisfiable whenever each finite subsystem of this system is satisfiable.

THEOREM 3. *A system is $\mathfrak{n}$-replete if and only if it is elementarily $\mathfrak{m}$-compact for every $\mathfrak{m} < \mathfrak{n}$.*

Proof. It is quite obvious that elementary $\mathfrak{m}$-compactness for every $\mathfrak{m} < \mathfrak{n}$ implies $\mathfrak{n}$-repleteness. The converse implication follows from Theorem A.2 of [10] (4). In fact, an equivalent of $\mathfrak{n}$-repleteness given in A.2 (condition (i)) easily implies $\mathfrak{m}$-compactness for every $\mathfrak{m} < \mathfrak{n}$.

On account of Theorem 3 we can translate several results announced in [10] and [12] into our terminology:

THEOREM 4. *Let $\mathfrak{R}_u$ $(u \in U)$ be a system of similar relational systems and $\boldsymbol{I}$ a prime ideal of subsets of U such that U is a union of $\aleph_0$ members of $\boldsymbol{I}$* (5). *Then the reduced product $\mathop{\mathfrak{P}}\limits_{u \in U} \mathfrak{R}_u/\boldsymbol{I}$ is elementarily $\aleph_0$-compact* (6).

THEOREM 5. *If $\mathfrak{R} = \langle A, \{R_t\}_{t \in T}\rangle$ is a system with $\aleph_0 \leqslant \overline{\overline{A}} \leqslant 2^{\mathfrak{m}}$ and $\overline{\overline{T}} \leqslant 2^{\mathfrak{m}}$, then $\mathfrak{R}$ has an elementary extension $\mathfrak{S}$, which is elementarily $\mathfrak{m}$-compact and $\overline{\overline{\mathfrak{S}}} = 2^{\aleph_\alpha}$* (7).

(4) The proof has not yet been published.

(5) i. e. $\boldsymbol{I}$ is not $\aleph_0$-additive. If $\overline{\overline{U}}$ is of 01-measure 0, as all but extremely large cardinals are — see [22], then every prime non principal ideal of subsets of U satisfies this supposition.

(6) A direct proof of Theorem 4 would be quite analogous to the proof of Theorem 1. A still more general result (on the assumption of some cases of the generalized continuum hypothesis) was announced in [10] (Theorem A.4).

(7) This is a form of Theorem 1 announced in [12].

THEOREM 6. *If* $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ *and* $\aleph_0 \leqslant \overline{\overline{\mathfrak{R}}} \leqslant \aleph_{\alpha+1}$ *then* $\mathfrak{R}$ *has an elementary extension* $\mathfrak{S}$, *which is elementarily* $\aleph_\alpha$*-compact and* $\overline{\overline{\mathfrak{S}}} = \aleph_{\alpha+1}$ (8).

THEOREM 7. *If* $\mathfrak{R} = \langle A, \{R_t\}_{t\in T}\rangle$ *is a system with* $\aleph_0 \leqslant \overline{\overline{A}}$ *and* $\overline{\overline{T}} \leqslant \aleph_\alpha$ *and* $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$, *then there is, up to an isomorphism, exactly one elementarily* $\aleph_\alpha$*-compact system* $\mathfrak{S}$ *elementarily equivalent to* $\mathfrak{R}$ *with* $\overline{\overline{\mathfrak{S}}} = \aleph_{\alpha+1}$ (9).

More informations on repleteness, the strictly related notions of universality and homogeneity and thus on elementary compactness of various relational systems and algebras are given in [10], [12], [13] and [20].

(8) This is a form of Corollary A.5 (a) announced in [10].

(9) This is a form of a theorem of Vaught, see [10], Theorem A.1, or [20], where analogous results are proved (Theorems 3.4, 3.5 and 3.6).

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