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LOCALLY COMPACT GROUPS AND THE CONTINUUM HYPOTHESIS

BY

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1. Introduction. Hulanicki [6] and Jones [7] (see 8.16, [5]) proved the following theorem:

THEOREM A. *If a locally compact group* (1) *has exactly* $\aleph_1$ *elements, then it is metrizable.*

Since any non-discrete locally compact group has at least $\mathfrak{c}$ elements (see 4.26, [5]), this amusing theorem is essentially vacuous if the continuum hypothesis were to fail so that $\aleph_1 < \mathfrak{c}$. It is natural to ask whether Theorem A can be proved with $\aleph_1$ replaced by $\mathfrak{c}$ without using the continuum hypothesis. However, it is easily seen that this cannot be done, at least without some additional hypothesis, since

If $2^{\aleph_1} = \mathfrak{c}$, *then there is a non-metrizable compact Abelian group having* $\mathfrak{c}$ *elements.*

To see this, consider the character group $G^{\wedge}$ of any discrete Abelian group G having $\aleph_1$ elements. By 2.1 below, $\mathrm{card}(G^{\wedge}) = 2^{\aleph_1} = \mathfrak{c}$ and, since G is not countable, $G^{\wedge}$ is not metrizable (see 2.2.6, [10]).

Hartman and Hulanicki [3] proved the following, assuming the generalized continuum hypothesis:

THEOREM B. *If* G *is a compact Abelian group and* $\mathrm{card}(G) \geqslant 2^{2^{\mathfrak{m}}}$ *for some cardinal* $\mathfrak{m}$, *then* G *contains a dense subset having* $\mathfrak{m}$ *elements.*

Again it is natural to enquire whether the continuum hypothesis is really needed. And again it is easy to see that it is. For example,

If there is a cardinal $\mathfrak{n} > \mathfrak{c}$ *such that* $2^{\mathfrak{n}} = 2^{\mathfrak{c}}$, *then there is a compact Abelian group with* $2^{2^{\aleph_0}}$ *elements that does not have a countable dense subset.*

In fact, let $G^{\wedge}$ be the character group of a discrete Abelian group G having $\mathfrak{n}$ elements. By theorem 2.1 below, $\mathrm{card}(G^{\wedge}) = 2^{\mathfrak{n}} = 2^{2^{\aleph_0}}$. Since $\mathrm{card}(G) > \mathfrak{c}$, $G^{\wedge}$ admits more than $\mathfrak{c}$ continuous functions and hence $G^{\wedge}$ cannot contain a countable dense subset.

(1) All topological groups are assumed to be Hausdorff.

Now let G be an Abelian group and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be locally compact group topologies on G such that $\mathcal{O}_1 \supset \mathcal{O}_2 \neq \mathcal{O}_1$.

THEOREM C. *There are at least* $2^{\aleph_1}$ $\mathcal{O}_1$*-continuous characters that are not* $\mathcal{O}_2$*-continuous.*

This is Theorem 3.3 of Hewitt's paper [4]; proofs have also been given by Glicksberg [2] and the author [9]. In view of Theorems A and B, I believed it is reasonable to expect that Theorem C cannot be proved with $2^{\mathfrak{c}}$ in place of $2^{\aleph_1}$ unless the continuum hypothesis is invoked. Nevertheless, Hewitt [4] conjectured that $2^{\aleph_1}$ could be replaced by $2^{\mathfrak{c}}$ without appealing to the continuum hypothesis. The purpose of this paper is to prove (in 2.5) that Hewitt's conjecture is true.

2. Hewitt's conjecture is true. The following theorem of Kakutani ([8] or 24.47, [5]) is vital in various computations of this paper. The symbol $\hat{}$ always designates a character group.

2.1. THEOREM (Kakutani). *Let G be an infinite discrete Abelian group. Then* $\operatorname{card}(G\hat{}\,) = 2^{\operatorname{card}(G)}$ (2).

To prove the lemmas below, we need the following facts about the real line R. Let m be a positive integer. Elements $(x_1, \ldots, x_m)$ of R^m will be written as $\boldsymbol{x}$. The character group of R^m is $\{\psi_{\boldsymbol{y}} : \boldsymbol{y} \in R^m\}$ where $\psi_{\boldsymbol{y}}(\boldsymbol{x}) = \exp[2\pi i(x_1 y_1 + \ldots + x_m y_m)]$. The character-group (or Pontryagin) topology for $\{\psi_{\boldsymbol{y}} : \boldsymbol{y} \in R^m\}$ can be seen to be that of R^m; i. e., $\boldsymbol{y} \to \psi_{\boldsymbol{y}}$ is a homeomorphism. For further details, we refer the reader to 23.27. f [5]. The following lemma is a simple consequence of the Pontryagin duality theorem and, of course, could be given in more generality.

2.2. LEMMA. *If $\mathscr{H}$ is a subgroup of $\{\psi_{\boldsymbol{y}} : \boldsymbol{y} \in R^m\}$ that separates points of R^m, then $\{\boldsymbol{y} \in R^m : \psi_{\boldsymbol{y}} \in \mathscr{H}\}$ is a dense subgroup of R^m.*

Proof. Let $A = \{\boldsymbol{x} \in R^m : \psi_{\boldsymbol{y}}(\boldsymbol{x}) = 1$ for all $\psi_{\boldsymbol{y}} \in \mathscr{H}\}$ and let $\mathscr{A} = \{\boldsymbol{y} \in R^m : \psi_{\boldsymbol{y}}(A) = 1\}$. Then 24.10 and 23.24.a of [5] imply that $\{\boldsymbol{y} \in R^m : \psi_{\boldsymbol{y}} \in \mathscr{H}\}^- = \mathscr{A}$. By hypothesis, we have $A = \{0\}$ so that $\mathscr{A} = R^m$. This proves the lemma.

2.3. LEMMA. *Suppose that G is a compact Abelian group and that, for some positive integer m and some compact Abelian group F, there is a continuous isomorphism φ mapping $R^m \times F$ onto a proper dense subgroup of G. Then* $\operatorname{card}\big(G/\varphi(R^m \times F)\big) \geqslant \mathfrak{c}$.

Proof. Case 1. Suppose that the factor F is trivial; then we regard φ as a mapping from R^m into G. Let $\mathscr{T}$ denote the topology on R^m under which φ is a topological isomorphism. Let $\mathscr{E}$ denote the group of characters that are continuous on $(R^m, \mathscr{T})$. Since $\mathscr{T}$ is weaker than the

(2) Of course, the group $G\hat{}$ is compact.

usual topology on R^m, it follows that $\mathscr{E} \subseteq \{\psi_{\boldsymbol{y}} : \boldsymbol{y} \in R^m\}$. Hence $\mathscr{E} = \{\psi_{\boldsymbol{y}} : \boldsymbol{y} \in E\}$ for some subgroup E of R^m. Since each continuous character on $\varphi(R^m)$ extends uniquely to a continuous character on G, it follows that $G^\wedge$ is isomorphic with $\mathscr{E}_d$ and hence with E_d (³). Since $\mathscr{T}$ is a Hausdorff topology, $\mathscr{E}$ separates points of R^m and hence, by 2.2, E is dense in R^m with the usual topology.

Now we define a mapping ν from $(R^m, \mathscr{T})$ into $E_d^\wedge$ as follows: $\nu(\boldsymbol{x})(\boldsymbol{y}) = \nu(\boldsymbol{x})(\psi_{\boldsymbol{y}}) = \psi_{\boldsymbol{y}}(\boldsymbol{x})$. Then clearly $\nu(R^m) = E^\wedge$ where $E^\wedge$ denotes the continuous characters on E regarded as a (dense) subgroup of R^m. As noted in 1.12, [1], ν is a topological isomorphism of $(R^m, \mathscr{T})$ onto a dense subgroup of $E_d^\wedge$. According to a theorem of Weil ([11] or 1.1, [1]) the compact group containing a dense copy of $(R^m, \mathscr{T})$ is unique up to a topological isomorphism leaving elements of the subgroup pointwise fixed. It follows that $E_d^\wedge / E^\wedge$ is isomorphic with $G/\varphi(R^m)$.

Thus it suffices to prove that $\mathrm{card}(E_d^\wedge / E^\wedge) \geqslant \mathfrak{c}$. Since E is dense in R^m, E contains a linearly independent subset $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ of R^m. Let $\boldsymbol{e}_k$ denote the element of R^m whose j-th coordinate is δ_{jk}. There is a one-to-one linear mapping of R^m onto R^m that maps $\boldsymbol{x}_k$ onto $\boldsymbol{e}_k$. This mapping is also a topological isomorphism of R^m onto R^m and hence it suffices to prove the result in the case that E contains $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$. In other words, we may assume that $Z^m \subseteq E$ where Z denotes the group of integers.

Let $\mathscr{A} = \{\varkappa \in E_d^\wedge : \varkappa(Z^m) = 1\}$. We first show that $E_d^\wedge = \mathscr{A} \cdot E^\wedge$. Consider any $\varkappa \in E_d^\wedge$. For each k, $\varkappa(\boldsymbol{e}_k) = \exp[2\pi i y_k]$ for some y_k. Then if $\boldsymbol{y} = (y_1, \ldots, y_m)$, we have $\psi_{\boldsymbol{y}}(\boldsymbol{e}_k) = \varkappa(\boldsymbol{e}_k)$ for each k. It follows that $\varkappa\psi_{\boldsymbol{y}}^{-1} \in \mathscr{A}$ and hence that $\varkappa = \varkappa\psi_{\boldsymbol{y}}^{-1} \cdot \psi_{\boldsymbol{y}} \in \mathscr{A} \cdot E^\wedge$. Now the first isomorphism theorem for groups states that $E_d^\wedge / E^\wedge = (\mathscr{A} \cdot E^\wedge)/E^\wedge$ is isomorphic with $\mathscr{A}/(\mathscr{A} \cap E^\wedge)$.

Therefore it now suffices to prove that $\mathrm{card}\big(\mathscr{A}/(\mathscr{A} \cap E^\wedge)\big) \geqslant \mathfrak{c}$. We will accomplish this by showing that $\mathrm{card}(\mathscr{A}) \geqslant \mathfrak{c}$ and that $\mathscr{A} \cap E^\wedge$ is countable. Since Z^m is not dense in R^m, E cannot be a union of a finite number of translates of Z^m. Hence E/Z^m is infinite. By 23.25, [5], $\mathscr{A}$ is isomorphic to the character group of E_d/Z^m and hence $\mathrm{card}(\mathscr{A}) \geqslant 2^{\aleph_0} = \mathfrak{c}$ by 2.1. Finally, $\mathscr{A} \cap E^\wedge$ is countable because $\mathscr{A} \cap E^\wedge \subseteq \{\psi_{\boldsymbol{y}} : \boldsymbol{y} \in Z^m\}$. Indeed, if $\psi_{\boldsymbol{y}} \in \mathscr{A} \cap E^\wedge$, then $\psi_{\boldsymbol{y}}(\boldsymbol{e}_k) = \exp[2\pi i y_k] = 1$ and hence $y_k \in Z$ for each k.

Case 2. We now consider the general case and hence we regard φ as a mapping from $R^m \times F$ into G. It is easy to see that $\varphi^\sim(\boldsymbol{x}) = \varphi(\boldsymbol{x}, e)_\varphi(F)$ defines a continuous isomorphism of R^m onto a proper dense subgroup of the compact group $G/\varphi(F)$. By case 1, we have $\mathrm{card}\big(G/\varphi(F)/\varphi^\sim(R^m)\big) \geqslant \mathfrak{c}$. Since $\varphi^\sim(R^m) = \varphi(R^m \times F)/\varphi(F)$, we have $\mathrm{card}\big(G/\varphi(F)/\varphi(R^m \times$

(³) For any group G, G_d denotes G with the discrete topology.

$\times F)/\varphi(F)) \geqslant \mathfrak{c}$. By virtue of the second isomorphism theorem for groups, we also have $\operatorname{card}\big(G/\varphi(R^m \times F)\big) \geqslant \mathfrak{c}$.

2.4. THEOREM. *Let G be an Abelian group with locally compact group topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ satisfying $\mathcal{O}_1 \supset \mathcal{O}_2 \neq \mathcal{O}_1$. Then* $\operatorname{card}(G, \mathcal{O}_1)^\wedge \geqslant 2^{\mathfrak{c}}$.

Proof. Let φ denote the continuous identity mapping of $(G, \mathcal{O}_1)$ onto $(G, \mathcal{O}_2)$. It is trivial that $(G, \mathcal{O}_2)$ contains a compactly generated open and closed subgroup. From the structure theorem for compactly generated Abelian groups (9.8, [5]), we see that there is an open and closed subgroup that is topologically isomorphic with $R^k \times G_0$, where G_0 is compact. Let J and K be subgroups of G such that $(J, \mathcal{O}_2)$ is topologically isomorphic with R^k, $(K, \mathcal{O}_2)$ is topologically isomorphic with G_0, $J \cap K = \{e\}$, and JK is $\mathcal{O}_2$-open. If φ were a homeomorphism when restricted to J and when restricted to K, then an elementary fact about products (6.12, [5]) would imply that φ is a homeomorphism on JK and hence a homeomorphism on G. Thus the hypotheses of this theorem hold for J or else for K. Because of this, it suffices to prove the theorem for the two cases:

1. $(G, \mathcal{O}_2)$ is topologically isomorphic with R^k,
2. $(G, \mathcal{O}_2)$ is compact.

Case 1. This follows immediately from Hewitt's Theorem 2.2, [4] which states that $(G, \mathcal{O}_1)$ is topologically isomorphic with $R^{k-l} \times R_d^l$ for some positive integer l.

Case 2. Just as with $(G, \mathcal{O}_2)$, $(G, \mathcal{O}_1)$ contains an open and closed subgroup H that is topologically isomorphic with $R^m \times F$, where F is compact. If $(G, \mathcal{O}_1)$ were σ-compact, 5.29, [5] would imply that φ is a homeomorphism. Hence we infer that G/H is uncountable.

Suppose that H is $\mathcal{O}_2$-closed. Then $(G/H, \mathcal{O}_2)$ is compact and by 4.26, [5], we have $\operatorname{card}(G/H) \geqslant \mathfrak{c}$. Suppose that H is not $\mathcal{O}_2$-closed. Then $m > 0$ and $(H, \mathcal{O}_2)$ is a continuous isomorphic image of $R^m \times F$. Replacing G by the $\mathcal{O}_2$-closure of H if necessary, we may assume that H is $\mathcal{O}_2$-dense in G. Consequently, 2.3 shows that $\operatorname{card}(G/H) \geqslant \mathfrak{c}$.

Thus whether H is $\mathcal{O}_2$-closed or not, we have $\operatorname{card}(G/H) \geqslant \mathfrak{c}$. Since H is $\mathcal{O}_1$-open, G/H is $\mathcal{O}_1$-discrete and a final application of 2.1 yields $\operatorname{card}(G, \mathcal{O}_1)^\wedge \geqslant \operatorname{card}(G/H, \mathcal{O}_1)^\wedge \geqslant 2^{\mathfrak{c}}$.

2.5. COROLLARY. *Let G, $\mathcal{O}_1$, and $\mathcal{O}_2$ be as in 2.4. Then there are $2^{\mathfrak{c}}$ $\mathcal{O}_1$-continuous characters that are not $\mathcal{O}_2$-continuous.*

This follows from the fact that $(G, \mathcal{O}_2)^\wedge$ is a proper subgroup of $(G, \mathcal{O}_1)^\wedge$; see Theorem C.

Added June 11, 1964. M. Rajagopalan has independently announced Corollary 2.5. See abstract 613-6, Notices of the American Mathematical Society 11 (1964), p. 443.

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