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NERVES AND SET-THEORETICAL INDEPENDENCE

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1. The *nerve* of a family of sets $F_1, \ldots, F_n$ is by definition (see e. g. [1], p. 201) a geometrical complex with vertices — we shall denote it for brevity's sake also by $F_1, \ldots, F_n$ — such that any simplex of the form $(F_{k_1}, \ldots, F_{k_t})$ belongs to this complex iff $F_{k_1} \cap \ldots \cap F_{k_t} \neq \emptyset$.

The sets $E_1, \ldots, E_n$ are called *independent* if no atom (i. e. the intersection of some of these of sets with the complements of some other of them) is void (see e. g. [2]).

The aim of this paper is to characterize the independence of sets in terms of the geometrical structure of their nerve.

Let X denote a fixed non-void set. For every system $E_1, \ldots, E_n$ of subsets of X we denote by $U(E_1, \ldots, E_n)$ the system

$$E_1, \ldots, E_n, E_{n+1}, \ldots, E_{2n},$$

where $E_{n+k} = E'_k = X \setminus E_k$ for $k = 1, \ldots, n$.

It is easy to see that

(i) *A necessary and sufficient condition of the independence of sets* $E_1, \ldots, E_n$ *is that each system* $j_1, \ldots, j_n$ *of indices* $1, \ldots, 2n$, *where no pair has the difference* n, *satisfies the inequality*

$$E_{j_1} \cap \ldots \cap E_{j_n} \neq \emptyset. \tag{1}$$

Moreover,

(ii) *For every system* $E_1, \ldots, E_n$ *of subsets of* X, *at least one system* $j_1, \ldots, j_n$ *of indices* $1, \ldots, 2n$, *where no pair has the difference* n, *satisfies inequality* (1).

This follows from (i) and from the fact that X is the union of all the atoms of the system $E_1, \ldots, E_n$.

(iii) *For any system* $E_1, \ldots, E_n$ *of subsets of* X *the nerve* N *of the system* $U(E_1, \ldots, E_n)$ *is of dimension* $n-1$.

Proof. Indeed, by (ii) there exists at least one system $j_1, \ldots, j_n$ of indices without pairs having the difference n and which satisfies (1).

Thus the $(n-1)$-dimensional simplex $(E_{j_1}, \ldots, E_{j_n})$ belongs to the nerve N by definition. Hence $\dim N \geqslant n-1$. On the other hand, $\dim N \leqslant n-1$, because no simplex with more than n vertices belongs to N as the intersection of more than n sets from $U(E_1, \ldots, E_n)$ is void. The proof is complete.

Let us now consider, in the Euclidean space R^n, the spherical surface S_{n-1} with the centre at the origin and with a system of perpendicular half-axes $L_1, \ldots, L_{2n}$ such that every union $L_k \cup L_{n+k}$ $(k = 1, \ldots, n)$ forms an axis. We denote by p_i the intersection point of L_i and S_{n-1}. Thus p_k and p_{n+k} are antipodic points. There exists a decomposition of S_{n-1} into 2^n spherical simplexes such that 1° their vertices belong to the set $\{p_1, \ldots, p_{2n}\}$ and 2° no simplex has a pair of antipodic vertices. These simplexes constitute evidently a triangulation of S_{n-1}. Thus the geometrical complex T consisting of this simplexes may be identified with S_{n-1}. Therefore

(iv) *The simplex $(p_{j_1}, \ldots, p_{j_n})$ belongs to the complex T iff the system $j_1, \ldots, j_n$ contains no pair of numbers with difference n.*

THEOREM. *The sets $E_1, \ldots, E_n$ are independent iff the nerve N of the system $U(E_1, \ldots, E_n)$ is homeomorphic with a spherical surface.*

Proof. Necessity. Let the set E_k correspond to the vertex p_k of the complex T. By (i) and (iv), T is the nerve of the system $U(E_1, \ldots, E_n)$. It remains to put $N = T$.

Sufficiency. Suppose that the sets $E_1, \ldots, E_n$ are dependent. Then there exists a system $j_1, \ldots, j_n$ of indices with values from among the numbers $1, \ldots, 2n$ with no pair of difference n such that $E_{j_1} \cap \ldots \cap E_{j_n} = \emptyset$. Thus the simplex $(E_{j_1}, \ldots, E_{j_n})$ does not belong to the nerve N.

It remains to let the set E_k correspond to the vertex p_k for $k = 1, 2, \ldots, n$ in order to see that the nerve N is then an $(n-1)$-dimensional proper subcomplex of T, and thus N is homeomorphic with a proper subset of the spherical surface of the same dimension.

By (iii) the Theorem may be modified as follows: *the sets $E_1, \ldots, E_n$ are independent iff the nerve N of the system $U(E_1, \ldots, E_n)$ is homeomorphic with S_{n-1}.*

2. It is known that the set-theoretical independence is a special case of the general algebraic independence, namely, it coincides with the independence in the algebra $\mathfrak{B} = \langle 2^X; \cup, ' \rangle$, where 2^X denotes the class of all subsets of X (see [2] and [3]).

Besides the algebra $\mathfrak{B}$, it is possible to take into consideration some "weaker" algebra, e. g. $\mathfrak{B}_1 = \langle 2^X; \cup, \setminus \rangle$. The independence of sets in $\mathfrak{B}_1$ does not give so nice geometrical interpretation as in the case of algebra $\mathfrak{B}$. It is possible to give only some necessary conditions, the geometrical sense of which is much more complicated.

Marczewski [2] has proved that the independence in algebra $\mathfrak{B}_1$ may be characterized as follows:

(v) *The sets* $E_1, \ldots, E_n$ *are independent in algebra* $\mathfrak{B}_1$ *iff for each system* $j_1, \ldots, j_n$ *of indices with* $1 \leqslant j_k \leqslant n$ *for* $k = 1, 2, \ldots, n$ *such that no pair of them has the difference equal to* n *and* $j_k > n$ *for at least one index, the relation*

$$E_{j_1} \cap \ldots \cap E_{j_n} \neq \emptyset$$

holds true.

Since the independence of sets in algebra $\mathfrak{B}$ implies their independence in algebra $\mathfrak{B}_1$, we may limit ourselves to consider only the sets that are dependent in algebra $\mathfrak{B}$.

We shall prove

(vi) *If the sets* $E_1, \ldots, E_n$ *are independent in algebra* $\mathfrak{B}_1$ *and dependent in algebra* $\mathfrak{B}$, *then the nerve* N *of the system* $U(E_1, \ldots, E_n)$ *is homeomorphic with an* $(n-1)$*-dimensional sphere* Q_{n-1}.

Proof. If the sets $E_1, \ldots, E_n$ are dependent in the algebra $\mathfrak{B}$, then we have $E_{n+1} \cap \ldots \cap E_{2n} = \emptyset$.

Now, if we let, for $k = 1, 2, \ldots, n$, the set E_k correspond to the vertex p_k of the complex T, then by (iv) the nerve N of the system $U(E_1, \ldots, E_n)$ is given by

$$N = T \setminus (p_{n+1}, \ldots, p_{2n}).$$

Therefore N is homeomorphic with Q_{n-1}.

Remark. The condition of N, being given by $N = T \setminus (p_{n+1}, \ldots, p_{2k})$, is not sufficient for the sets $E_1, \ldots, E_n$ to be independent in algebra $\mathfrak{B}_1$. Indeed, to see this it is sufficient to consider two sets A and B such that $A \cup B = X$ and $A \cap B \neq \emptyset$.

REFERENCES

[1] C. Kuratowski, *Topologie I*, Warszawa 1958.

[2] E. Marczewski, *Independence in algebras of sets and Boolean algebras*, Fundamenta Mathematicae 48 (1960), p. 135-140.

[3] — *Independence and homomorphisms in abstract algebras*, ibidem 50 (1961), p. 45-61.

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