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ON ISOMETRIC DOMAINS OF POSITIVE OPERATORS ON L^p-SPACES

BY

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1. Introduction and preliminaries. In this paper*, $(X, \mathscr{A}, m)$ and $(Y, \mathscr{B}, n)$ stand for σ-finite measure spaces. For $1 < p < \infty$ we denote by $L^p(m) = L^p(X, \mathscr{A}, m)$ the Banach lattice of all p-integrable real-valued functions on X with the standard norm $\|\cdot\|_p$ (also written as $\|\cdot\|$ when no confusion can arise) and order. In the case $X = [0, 1]$, where $\mathscr{A}$ is the σ-algebra of Lebesgue measurable sets and $m = \lambda$ (the Lebesgue measure on $\mathscr{A}$), we write briefly L^p instead of $L^p(m)$.

By $\mathscr{L}(L^p(m), L^r(n))$ we denote the Banach space of all bounded linear operators from $L^p(m)$ into $L^r(n)$ and by $\mathscr{L}_+(L^p(m), L^r(n))$ its subset consisting of all positive operators, i.e., $T \in \mathscr{L}_+(L^p(m), L^r(n))$ if and only if $Tf \geqslant 0$ whenever $f \geqslant 0$.

Given an operator $T \in \mathscr{L}(L^p(m), L^r(n))$, we define its isometric domain as

$$M(T) = \{f \in L^p(m): \|Tf\| = \|T\| \|f\|\}.$$

Thus, for $f \in L^p(m)$ with $f \neq 0$, we have $f \in M(T)$ if and only if T attains its norm at f. Moreover, if $M(T) \neq \{0\}$, then T is said to be *norm attaining*. It follows from a result of Lindenstrauss ([7], Theorem 1) that the set of norm attaining operators is norm dense in $\mathscr{L}(L^p(m), L^r(n))$. On the other hand, not every (positive) operator is norm attaining, e.g., $T \in \mathscr{L}(L^p)$ defined by $(Tf)(x) = xf(x)$.

The main result of the present paper asserts that $M(T)$ is a closed linear sublattice of $L^p(m)$ for every $T \in \mathscr{L}_+(L^p(m), L^p(n))$ (Theorem 2 in Section 2). This is no more true, in general, if $p \neq r$ (Proposition 2 in Section 3). In this connection we introduce in Section 4 the class of elementary operators for which the assertion of Theorem 2 still holds if $r \leqslant p$ (Theorem 4). It is worthwhile to mention that the proof of Theorem 2 consists in a reduction to the case where $X = Y = [0, 1]$, $m = n = \lambda$, and T is doubly stochastic. This case is settled with the help of some results of Ryff [11], [12].

* This paper is based on a part of the author's doctoral thesis written under the supervision of Professor Anzelm Iwanik.

In the sequel of this section we introduce some more notation and establish a number of auxiliary results. We define the *support* of $f \in L^p(m)$, denoted by $\operatorname{supp} f$, to be the set $\{x: f(x) \neq 0\}$. We write $f \perp g$ provided $f, g \in L^p(m)$ are orthogonal, that is, have disjoint supports.

(*) *Let* $f \in L^p_+(m)$, $g \in L^p(m)$, *and let* $T \in \mathscr{L}_+(L^p(m), L^r(n))$. *If* $\operatorname{supp} g \subset \operatorname{supp} f$, *then* $\operatorname{supp} Tg \subset \operatorname{supp} Tf$.

Indeed, we have $|g| \wedge nf \uparrow |g|$, and so $T(|g| \wedge nf) \uparrow T|g|$. Hence

$$\operatorname{supp} Tg \subset \operatorname{supp} T|g| = \bigcup_{n=1}^{\infty} \operatorname{supp} T(|g| \wedge nf) \subset \operatorname{supp} Tf.$$

LEMMA 1. *Let* $p \leqslant r$, $T \in \mathscr{L}_+(L^p(m), L^r(n))$, *and let* $f \in M(T)$. *If* $f_1, f_2 \in L^p(m)$, $f_1 \perp f_2$, *and* $Tf_1 \perp Tf_2$, *then* $f_1, f_2 \in M(T)$. *If, in addition,* $T \neq 0$ *and* $p < r$, *then* $f_1 = 0$ *or* $f_2 = 0$.

Proof. We may and do assume that $\|T\| = 1$ and $\|f\| = 1$. Then

$$1 = \|Tf\|_r^r = \|Tf_1\|_r^r + \|Tf_2\|_r^r \leqslant \|Tf_1\|_r^p + \|Tf_2\|_r^p = \|f_1\|_p^p + \|f_2\|_p^p = 1.$$

Hence

$$\|Tf_i\|_r^r = \|Tf_i\|_r^p = \|f_i\|_p^p \quad \text{for } i = 1, 2. \tag{1}$$

If $T \in \mathscr{L}_+(L^p(m), L^r(n))$ and $f \in M(T)$, then

$$|f| \in M(T), \tag{2}$$

$$|Tf| = T|f|, \quad Tf_+ \perp Tf_-, \tag{3}$$

$$(Tf)_+ = Tf_+, \quad (Tf)_- = Tf_-,$$

$$f_+, f_- \in M(T) \quad \text{if } p \leqslant r. \tag{4}$$

Indeed, (2) and the first equality of (3) follow from the inequality $|Tf| \leqslant T|f|$. Since

$$\|Tf_+ + Tf_-\| = \big\|T|f|\big\| = \|Tf\| = \|Tf_+ - Tf_-\|,$$

we have $Tf_+ \perp Tf_-$. It follows that $(Tf)_+ = (Tf_+ - Tf_-)_+ = Tf_+$. Finally, (4) is a consequence of Lemma 1 and (3).

We use the well-known identification of the conjugate space $[L^p(m)]^*$ with $L^{p'}(m)$, where $1/p' + 1/p = 1$, by the formula $\langle f, h\rangle = \int fh\,dm$ for $f \in L^p(m)$ and $h \in L^{p'}(m)$.

For $f \in L^p(m)$ we put

$$f^{p-1}(x) = |f(x)|^{p-1} \operatorname{sign} f(x).$$

We have $\|f^{p-1}\|_{p'}^{p'} = \langle f, f^{p-1}\rangle = \|f\|_p^p$. It follows that $f^{p-1} \in L^{p'}(m)$ and the corresponding functional attains its norm at f.

Consequently, if $T \in \mathscr{L}(L^p(m), L^r(n))$ attains its norm at f and $\|T\| = \|f\| = 1$, then $1 = \langle Tf, g^{r-1}\rangle = \langle f, T^* g^{r-1}\rangle$, where $g = Tf$. From the strict convexity of $L^{p'}(m)$ we get $T^* g^{r-1} = f^{p-1}$, that is,

(5) $$T^*(Tf)^{r-1} = f^{p-1}.$$

By a similar argument and (3), for $T \geqslant 0$ we have

(6) $$T^* |Tf|^{r-1} = |f|^{p-1}.$$

Further, for $A \in \mathscr{A}$ we define an operator I_A on $L^p(m)$ by the formula $I_A f = 1_A f$.

LEMMA 2. *Let* $f \in L^p_+(m)$ *and* $T \in \mathscr{L}_+(L^p(m), L^r(n))$ *be such that* $\operatorname{supp} T^*(Tf)^{r-1} \subset \operatorname{supp} f$. *Then*

$$T = I_{\operatorname{supp} Tf}\, T I_{\operatorname{supp} f} + I_{(\operatorname{supp} Tf)^c}\, T I_{(\operatorname{supp} f)^c}.$$

In particular, $g \perp f$ *implies* $Tg \perp Tf$ *for every* $g \in L^p(m)$.

Proof. Put $A = \operatorname{supp} f$ and $B = \operatorname{supp} Tf$. We have to show that the operators $T_1 = I_{B^c} T I_A$ and $T_2 = I_B T I_{A^c}$ equal 0. Since, by (*), $\operatorname{supp} T I_A g \subset B$ for every $g \in L^p_+(m)$, the first assertion follows.

We show that $T_2^* = I_{A^c} T^* I_B = 0$. Fix $h \in L^{r'}_+(n)$. We have $\operatorname{supp} 1_B h \subset \operatorname{supp}(Tf)^{r-1}$. Hence, by assumption and (*), $\operatorname{supp} T^* I_B h \subset A$. Thus $T_2^* h = 0$.

Lemma 2 yields, in view of (2) and (5), the following:

(**) *If* $T \in \mathscr{L}_+(L^p(m), L^r(n))$, *then* $g \perp f$ *implies* $Tg \perp Tf$ *for every* $f \in M(T)$ *and* $g \in L^p(m)$.

Finally, for an operator $T \in \mathscr{L}(L^p(m), L^r(n))$ we put

$$J(T) = \{\operatorname{supp} f \colon f \in M(T)\}.$$

LEMMA 3. *Let* $T \in \mathscr{L}_+(L^p(m), L^p(n))$. *If* $A \in J(T)$ *and* $f \in M(T)$, *then* $1_A f$, $1_{A^c} f \in M(T)$.

Proof. Choose $u \in M(T)$ with $\operatorname{supp} u = A$. By (**) and (2), $T(1_{A^c} f) \perp T|u|$. Hence, by (*), we have $T(1_{A^c} f) \perp T(1_A |f|)$. This yields $T(1_{A^c} f) \perp T(1_A f)$, which, together with Lemma 1, implies the assertion.

2. The case $p = r$. We start with the following

THEOREM 1. *If* $T \in \mathscr{L}_+(L^p(m), L^p(n))$, *then* $J(T)$ *is a* σ-*ring of sets.*

Proof. We may and do assume that $\|T\| = 1$. It is sufficient to show that $J(T)$ is closed under differences and countable disjoint unions. If $f, u \in M(T)$, then by Lemma 3 we have $1_{(\operatorname{supp} u)^c} f \in M(T)$, and so $(\operatorname{supp} f) \setminus (\operatorname{supp} u) \in J(T)$. Now, let $A_n \in J(T)$ be pairwise disjoint and let $f_n \in M(T)$ be such that $\|f_n\| = 1$ and $\operatorname{supp} f_n = A_n$. By (**), $\operatorname{supp} Tf_n$ are pairwise disjoint. Putting

$$f = \sum_{n=1}^{\infty} 2^{-n/p} f_n$$

we have

$$\operatorname{supp} f = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \|Tf\| = \sum_{n=1}^{\infty} 2^{-n} \|Tf_n\|^p = 1 = \|f\|.$$

Remark 1. *If $T \in \mathscr{L}_+(L^p(m), L^p(n))$, then the σ-rings $J(T)$ and $J(T^*)$ are isomorphic.*

Indeed, assume that $\|T\| = 1$ and put $\phi(\operatorname{supp} f) = \operatorname{supp} Tf$ for $f \in M(T)$. By (*), (2) and (3), ϕ is well defined. Assume that $\|f\| = 1$. Since $\operatorname{supp} Tf = \operatorname{supp}(Tf)^{p-1}$, it follows from (5) that $\phi(\operatorname{supp} f) \in M(T^*)$. Arguing as in the proof of Theorem 1, we can show that if $A_n \in J(T)$ are pairwise disjoint, then so are $\phi(A_n)$ and $\phi(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \phi(A_n)$. Hence

$$\phi(A \cup B) = \phi(A \setminus B) \cup \phi(A \cap B) \cup \phi(B \setminus A) = \phi(A) \cup \phi(B),$$

$$\phi(A \setminus B) = \phi(A) \setminus \phi(B \cap A) = \phi(A) \setminus \big(\phi(B \cap A) \cup \phi(B \setminus A)\big) = \phi(A) \setminus \phi(B).$$

It follows that ϕ is a σ-homomorphism.

To show that ϕ maps $J(T)$ onto $J(T^*)$, fix $g \in M(T^*)$ with $\|g\| = 1$. Since, by (5), $T(T^*g)^{p-1} = g^{p-1}$, we have

$$(T^*g)^{p-1} \in M(T) \quad \text{and} \quad \phi\big(\operatorname{supp}(T^*g)^{p-1}\big) = \operatorname{supp} g.$$

Finally, if $\operatorname{supp} Tf = \emptyset$ and $f \in M(T)$, then $\operatorname{supp} f = \emptyset$. It follows that ϕ is a one-to-one mapping.

In the sequel of this section we shall need some notation and results of Ryff [11], [12] which we now recall.

For a measurable function $f\colon [0, 1] \to R$ we define a function $f^*\colon [0, 1] \to R$ by

$$f^*(x) = \sup\{y\colon \lambda(\{z\colon f(z) > y\}) > x\}.$$

The function f^* is non-increasing, right-continuous and has the same distribution as f. We have $\|f\|_p = \|f^*\|_p$ for every $f \in L^p$. It is also easy to see that $(f+1)^* = f^* + 1$.

Ryff introduced in [11] a partial ordering $\prec$ for functions $f, g \in L^1$ by defining $g \prec f$ whenever

$$\int_0^s g^*\, d\lambda \leqslant \int_0^s f^*\, d\lambda \quad \text{for } 0 \leqslant s < 1$$

and

$$\int_0^1 g^*\, d\lambda = \int_0^1 f^*\, d\lambda.$$

An operator $P \in \mathscr{L}_+(L^1)$ is said to be *doubly stochastic* provided $P1 = 1$ and $P^*1 = 1$. An operator $T \in \mathscr{L}(L^1)$ is doubly stochastic if and only if

$Tf \prec f$ for every $f \in L^1$ ([11], p. 1384–1385). Moreover, for $f, g \in L^1$ we have $g \prec f$ if and only if there exists a doubly stochastic operator $P \in \mathscr{L}(L^1)$ such that $g = Pf$ ([12], Theorem 3).

LEMMA 4. *Let $f, g \in L^p$. If $g \prec f$, then $\|g\|_p \leqslant \|f\|_p$. If, moreover, $\|g\|_p = \|f\|_p$, then $g^* = f^*$.*

Proof. Let P be a doubly stochastic operator such that $g = Pf$. Now, the first assertion follows from the fact that P when restricted to L^p is a contraction into L^p (M. Riesz Theorem; see [13], V.8.2).

Since $g^{**} = g^*$ and $f^{**} = f^*$, we have

$$\|g^*\|_p \leqslant \left\|\frac{g^* + f^*}{2}\right\|_p \leqslant \|f^*\|_p.$$

If $\|g\|_p = \|f\|_p$, it follows from the strict convexity of L^p that $g^* = f^*$.

LEMMA 5. *Let $\varphi \in L^p$ and let P be a doubly stochastic operator. If $\|P\varphi\|_p = \|\varphi\|_p$, then $\|P\varphi + 1\|_p = \|\varphi + 1\|_p$.*

Proof. By Lemma 4, we have $(P\varphi)^* = \varphi^*$. Hence $(P\varphi + 1)^* = (\varphi + 1)^*$, and so

$$\|P\varphi + 1\|_p = \|(P\varphi + 1)^*\|_p = \|(\varphi + 1)^*\|_p = \|\varphi + 1\|_p.$$

We note that since the results of Ryff used above are valid for an arbitrary finite measure space (see, e.g., [9], Theorems 1 and 3), Lemmas 4 and 5 are also valid in that setting. However, this generality is not needed for our purposes.

LEMMA 6. *If $T \in \mathscr{L}_+(L^p)$ and $f, u \in M(T)$, then $f + u \in M(T)$.*

Proof. We may and do assume that $\|T\| = 1$.

We first prove the assertion under the assumption that $\operatorname{supp} u \subset \operatorname{supp} f$ and $f \geqslant 0$. We also assume, without loss of generality, that $\|f\| = 1$.

Let τ: $[0, 1] \to [0, 1]$ be a Borel isomorphism such that

$$\lambda(C) = \int_{\tau^{-1}(C)} |f|^p \, d\lambda$$

for every Borel set $C \subset [0, 1]$ (see [14], 4.1 (i); cf. also [10], Chapter 15, Theorem 9). Putting $Uh(x) = f(x) h(\tau(x))$, we define a positive L^p-isometry U such that $U(L^p) = \{k \in L^p\colon \operatorname{supp} k \subset \operatorname{supp} f\}$ and $U1 = f$ (cf. [10], Chapter 15, Theorem 16).

Put $g = Tf$. By a similar argument, there exists a positive L^p-isometry V of $\{k \in L^p\colon \operatorname{supp} k \subset \operatorname{supp} g\}$ onto L^p such that $Vg = 1$. Since, in view of $(*)$, $\operatorname{supp} TUh \subset \operatorname{supp} g$ for $h \in L^p$, we may define an operator P by the formula $P = VTU$. Clearly, $P \geqslant 0$ and $P1 = 1$. Moreover, applying (5) three times, we get

$$P^* 1 = U^* T^* V^* 1 = U^* T^* g^{p-1} = U^* f^{p-1} = 1.$$

It follows that P can be (uniquely) extended to a doubly stochastic operator on L^1, which we still denote by P.

Put $\varphi = U^{-1}u$. We have $\|P\varphi\| = \|VTu\| = \|Tu\| = \|u\| = \|\varphi\|$. Hence, by Lemma 5, $\|P\varphi+1\| = \|\varphi+1\|$. On the other hand,

$$\|\varphi+1\| = \|U(\varphi+1)\| = \|u+f\|,$$

$$\|P\varphi+1\| = \|VT(u+f)\| = \|T(u+f)\|.$$

It follows that $f+u \in M(T)$.

In the general case put $A = \operatorname{supp} f_+$ and $B = \operatorname{supp} f_-$. By (4) we have $f_+,\ f_- \in M(T)$. Hence Lemma 3 yields $1_A u,\ 1_B u \in M(T)$. Consequently, $f_+ + 1_A u \in M(T)$ and $f_- - 1_B u \in M(T)$. Hence, by (**) we conclude that $T(f_+ + 1_A u) \perp T(f_- - 1_B u)$. Therefore

$$\|T(f+u)\|^p =$$

$$\|T(f_+ + 1_A u)\|^p + \|T(f_- - 1_B u)\|^p = \|f_+ + 1_A u\|^p + \|f_- - 1_B u\|^p = \|f+u\|^p.$$

THEOREM 2. *If* $T \in \mathscr{L}_+(L^p(m), L^p(n))$, *then* $M(T)$ *is a closed linear sublattice of* $L^p(m)$.

Proof. Clearly, $M(T)$ is closed. By (2), if $f \in M(T)$, then $|f| \in M(T)$. Obviously, $af \in M(T)$ for $a \in R$. Thus it is sufficient to show that if $f,\ u \in M(T)$, then $f+u \in M(T)$.

Assume, without loss of generality, that $\|T\| = 1$. Denote by $\tilde{\mathscr{A}}$ the smallest σ-algebra with respect to which f and u are measurable and put $\tilde{m} = m|\tilde{\mathscr{A}}$. Then $L^p(\tilde{m})$ is separable, and so there exists a σ-subalgebra $\tilde{\mathscr{B}}$ of $\mathscr{B}$ such that $L^p(\tilde{n})$ is separable and $T(L^p(\tilde{m})) \subset L^p(\tilde{n})$ with $\tilde{n} = n|\tilde{\mathscr{B}}$. By a well-known theorem there exist positive isometries S_1: $L^p(\tilde{n}) \to L^p$ and S_2: $L^{p'}(\tilde{m}) \to L^p$, where $1/p + 1/p' = 1$ (cf. [10], Chapter 15, Theorem 2). Then $T_0 = S_1 T S_2^*$ is a positive contraction. Put

$$\tilde{f} = S_2(f^{p-1})^{p'-1} \quad \text{and} \quad \tilde{u} = S_2(u^{p-1})^{p'-1}.$$

By (5), $S_2^*\tilde{f} = f$ and $S_2^*\tilde{u} = u$, whence $\tilde{f},\ \tilde{u} \in M(T_0)$. Therefore, by Lemma 6,

$$\|T(f+u)\| = \|S_1 T(f+u)\| = \|T_0(\tilde{f}+\tilde{u})\| = \|\tilde{f}+\tilde{u}\| \geqslant \|S_2^*(\tilde{f}+\tilde{u})\| = \|f+u\|.$$

Remark 2. In view of Theorem 2, $M(T)$ is a Banach lattice. Moreover, by (3), $T|f| = |Tf|$ for $f \in M(T)$. It follows that T, when restricted to $M(T)$, is a Banach-lattice isomorphism into $L^p(n)$ provided $T \neq 0$.

Recall that a finite measure m on $\mathscr{A}$ is said to be *perfect* provided for every measurable function f: $X \to R$ and $H \subset R$ such that $f^{-1}(H) \in \mathscr{A}$ there exists a Borel set $C \subset H$ such that $m(f^{-1}(C)) = m(f^{-1}(H))$ (see [8] for other equivalent conditions).

The following corollary is closely related to recent results of Deland and Shiflett ([3], Corollary 3), and Hardin ([5], Corollary 4.3):

COROLLARY. *Let m be perfect and assume that $T \in \mathscr{L}_+(L^p(m))$ has the norm* 1. *If* $1, f \in M(T)$ *with f being a one-to-one function, then T is an isometry. If, moreover,* $T1 = 1$ *and* $Tf = f$, *then T is the identity operator.*

Proof. Let K be the closed linear sublattice of $L^p(m)$ generated by 1 and f. By [13], Proposition III.11.2 (cf. also [6], Proposition), there exists a σ-subalgebra $\mathscr{A}'$ of $\mathscr{A}$ such that

$$K = \{f \in L^p(m) \colon f \text{ is } \mathscr{A}'\text{-measurable}\}.$$

On the other hand, since m is perfect and f is one-to-one, for every $A \in \mathscr{A}$ there exists a Borel set $C \subset R$ such that $m(A \Delta f^{-1}(C)) = 0$. From Theorem 2 it follows that $K = L^p(m)$. In particular, $M(T) = L^p(m)$, and so T is an isometry. The second part of the assertion is now a consequence of Remark 2.

Remark 3. As was pointed out by Professor F. Altomate, under the hypotheses of the Corollary, if $1, f \in M(T)$ and f is one-to-one, then $\{1, f\}$ is a Korovkin system in $L^p(m)$ with respect to sequences of positive contractions (i.e., for every sequence $(L_n)_{n \in N}$ of positive contractions on $L^p(m)$ such that $\lim_{n \to \infty} L_n(h) = h$ for all $h \in \{1, f\}$, we have $\lim_{n \to \infty} L_n(g) = g$ for all $g \in L^p(m)$).

In fact, as showed previously, the closed linear sublattice of $L^p(m)$ generated by $\{1, f\}$ coincides with $L^p(m)$, which implies (in fact, is equivalent to) that $\{1, f\}$ is a Korovkin system for positive contractions because of a result of Berens and Lorentz (see [1], p. 27, and [2], Theorem 2).

3. The case $p \neq r$. For $T \in \mathscr{L}_+(L^p(m), L^p(n))$ we define the *support* of T, denoted by $\operatorname{supp} T$, to be the smallest set $A \in \mathscr{A}$ (modulo m-null sets) such that $T1_A = T$. The existence of $\operatorname{supp} T$ follows from the σ-finiteness of m.

As easily seen from $(*)$, $f \in L^p_+(m)$ and $Tf = 0$ imply

$$(\operatorname{supp} f) \cap (\operatorname{supp} T) = \emptyset.$$

PROPOSITION 1. *Let* $1 < r \leqslant p < \infty$ *and let* $T \in \mathscr{L}_+(L^p(m), L^r(n))$. *If* $T^*(Tf)^{r-1} = f^{p-1}$ *for a certain function* $f \in L^p_+(m)$ *such that* $\|f\|_p = 1$ *and* $\operatorname{supp} f = \operatorname{supp} T$, *then* $\|T\| = 1$.

Proof. We have

$$1 = \langle f, T^*(Tf)^{r-1} \rangle = \langle Tf, (Tf)^{r-1} \rangle = \|Tf\|_r^r,$$

whence $\|T\| \geqslant 1$.

To prove the other inequality, consider $g \in L^p(m)$ of the form

$$g = \sum_{i=1}^{k} c_i 1_{A_i} f + g_1,$$

where $c_i \in R$, $\{A_i\}_{i=1}^k$ is a measurable partition of $\operatorname{supp} f$ and $g_1 \perp f$. We have

$$|Tg|^r = |T(\sum c_i 1_{A_i} f)|^r = (Tf)^r \left|\sum \frac{T(1_{A_i} f)}{Tf} c_i\right|^r$$
$$\leqslant (Tf)^r \sum \frac{T(1_{A_i} f)}{Tf} |c_i|^r = (Tf)^{r-1} T(\sum |c_i|^r 1_{A_i} f).$$

Hence, by assumption,

$$\|Tg\|_r^r \leqslant \int (Tf)^{r-1} T(\sum |c_i|^r 1_{A_i} f)\, dn$$
$$= \int (\sum |c_i|^r 1_{A_i} f)\, T^*(Tf)^{r-1}\, dm = \int \sum |c_i|^r 1_{A_i} f^p\, dm \leqslant \int |g|^r f^{p-r}\, dm.$$

Since the functions g defined above form a dense subset of $L^p(m)$, we have $\|Tg\|_r^r \leqslant \int |g|^r f^{p-r}\, dm$ for all $g \in L^p(m)$. In the case $p = r$, this immediately implies $\|T\| \leqslant 1$. In the case $p > r$, we apply, moreover, the estimate

$$\int |g|^r f^{p-r}\, dm \leqslant \left[\int (|g|^r)^{p/r}\, dm\right]^{r/p} \left[\int (f^{p-r})^{p/(p-r)}\, dm\right]^{(p-r)/p} = \|g\|_p^r, \tag{7}$$

which is a consequence of Hölder's inequality.

THEOREM 3. *Let* $1 < r < p < \infty$ *and let* $T \in \mathscr{L}_+(L^p(m), L^r(n))$. *Then* $J(T) \subset \{\emptyset, \operatorname{supp} T\}$ *and there exists at most one positive function f with norm 1 in $M(T)$. Moreover, if $g \in M(T)$, then $|g| = cf$, where c is a non-negative number.*

Proof. Assume, without loss of generality, that $\|T\| = 1$. Let $A \in J(T)$ and $m(A) > 0$. Then, by (2), there exists $f \in M(T)$ with norm 1 such that $f \geqslant 0$ and $\operatorname{supp} f = A$. We claim that if $h \in L^p(m)$, $\|h\|_p = 1$, and $h \perp f$, then $Th = 0$. Indeed, we have $\|Tf + aTh\|_r \leqslant \|f + ah\|_p$. Since, by (**), $Th \perp Tf$, we obtain

$$1 + a^r \|Th\|_r^r \leqslant (1 + a^p)^{r/p}.$$

Hence

$$\|Th\|_r^r \leqslant \frac{(1 + a^p)^{r/p} - 1}{a^r},$$

which yields $\|Th\|_r = 0$ when $a \downarrow 0$. Thus we have proved that $\operatorname{supp} T = A$.

To prove the second assertion take $g \in M(T)$. Then in (7) we have the equality. Hence there exists $c \in R_+$ with $|g| = cf$ ([10], Chapter 6, Theorem 2).

The following simple example contrasts with Theorems 2 and 3. It shows that if $1 < p < r$, then $M(T)$ need not be a linear space nor have any of the properties asserted in Theorem 3.

Example. Let $I \in \mathscr{L}(l_2^p, l_2^r)$ be the identity operator. If $p < r$, then $\|I\| = 1$ and

$$M(I) = \{(\alpha, 0), (0, \alpha)\colon \alpha \in R\}.$$

Our next result shows that Theorem 2 cannot be generalized in a certain direction (see [4] for another result of the same type).

PROPOSITION 2. *Suppose that* $1 < p < \infty$, $1 < r < \infty$, *and assume that* $T \in \mathscr{L}_+(L^p(m), L^r(n))$ *and* $\dim \operatorname{lin} M(T) \geqslant 2$. *Then* $M(T)$ *is a linear subspace of* $L^p(m)$ *if and only if* $p = r$.

Proof. The "if" part follows from Theorem 2.

Suppose $r < p$. Then it follows from Theorem 3 that there exist two linearly independent functions $f_1, f_2 \in M(T)$ such that $|f_1| = |f_2|$. Then $\emptyset \neq \operatorname{supp}(f_1 + f_2) \neq \operatorname{supp} T$. Hence, by Theorem 3, $f_1 + f_2 \notin M(T)$, and so $M(T)$ is not a linear space.

Suppose $r > p$. Observe that if $f \in M(T)$, then $f \geqslant 0$ or $f \leqslant 0$. Indeed, by (3), $Tf_+ \perp Tf_-$. Hence, in virtue of Lemma 1, $f_- = 0$ or $f_+ = 0$. Now, if $f_1, f_2 \in M(T)$ are linearly independent, there exists $\alpha \in R$ such that $(f_1 + \alpha f_2)_+ \neq 0$ and $(f_1 + \alpha f_2)_- \neq 0$. Consequently, $f_1 + \alpha f_2 \notin M(T)$, and so $M(T)$ is not a linear space.

4. The case of an elementary operator. We say that an operator $T \in \mathscr{L}_+(L^p(m), L^r(n))$ is *elementary* provided there are no non-zero operators $T_1, T_2 \in \mathscr{L}_+(L^p(m), L^r(n))$ such that $T = T_1 + T_2$ and

$$(\operatorname{supp} T_1) \cap (\operatorname{supp} T_2) = (\operatorname{supp} T_1^*) \cap (\operatorname{supp} T_2^*) = \emptyset.$$

THEOREM 4. *Let* $1 < r \leqslant p < \infty$ *and let* $T \in \mathscr{L}_+(L^p(m), L^r(n))$ *be an elementary operator. Then* $M(T)$ *is a linear sublattice of* $L^p(m)$ *with* $\dim M(T) \leqslant 1$. *Moreover, if* $f \in M(T)$ *and* $f \neq 0$, *then* $\operatorname{supp} f = \operatorname{supp} T$.

Proof. Case $r < p$. Suppose that there exist two linearly independent functions in $M(T)$. Then it follows from Theorem 3 that there is an $f \in M(T)$ such that $f_+ \neq 0$ and $f_- \neq 0$. Moreover, $\operatorname{supp} f = \operatorname{supp} T$. By (3), $\operatorname{supp} Tf_+$ and $\operatorname{supp} Tf_-$ are disjoint. From (*) we obtain

$$I_{(\operatorname{supp} Tf_+)^c} T I_{\operatorname{supp} f_+} = 0 \quad \text{and} \quad I_{(\operatorname{supp} Tf_-)^c} T I_{\operatorname{supp} f_-} = 0.$$

Since $\operatorname{supp} T = (\operatorname{supp} f_+) \cup (\operatorname{supp} f_-)$, we get

$$T = I_{\operatorname{supp} Tf_+} T I_{\operatorname{supp} f_+} + I_{\operatorname{supp} Tf_-} T I_{\operatorname{supp} f_-}.$$

Hence T is non-elementary, a contradiction. The second assertion follows from Theorem 3.

Case $r = p$. We start with the second assertion. As easily seen, $\operatorname{supp} f \subset \operatorname{supp} T$. Suppose, to get a contradiction, that the set $A = (\operatorname{supp} T) \setminus (\operatorname{supp} f)$ has a positive m-measure. We may and do assume that $\|T\| = \|f\| = 1$ and $f \geqslant 0$ (see (2)). Put

$$T_1 = I_{\operatorname{supp} Tf} T I_{\operatorname{supp} f} \quad \text{and} \quad T_2 = I_{(\operatorname{supp} Tf)^c} T I_{(\operatorname{supp} f)^c}.$$

Then $T_1 \neq 0$ since $T_1 f = Tf \neq 0$, and $T_2 \neq 0$ since $m(A) > 0$. Moreover, by

(*) and (**), $T = T_1 + T_2$, which contradicts the assumption that T is elementary.

Suppose now that $\dim M(T) \geqslant 2$ and let $f, g \in M(T)$ be linearly independent. Then there exists $a \in R$ such that $(f+ag)_+ \neq 0$ and $(f+ag)_- \neq 0$. As, by Theorem 2, $(f+ag)_+ \in M(T)$ and $(f+ag)_- \in M(T)$, we have obtained a contradiction with the second assertion of the theorem.

Remark 4. *If $T \in \mathscr{L}_+(L^p(m), L^p(n))$ and A is an atom of the σ-ring $J(T)$, then TI_A is an elementary operator.*

Indeed, suppose on the contrary that $TI_A = T_1 + T_2$, where $T_1 = I_C TI_B$ and $T_2 = I_{C^c} TI_{B^c}$ are non-zero. This implies that $m(A \cap B) > 0$ and $m(A \cap B^c) > 0$. Choose $f \in M(T)$ with $\operatorname{supp} f = A$. Then, by Lemma 1, we obtain $1_{A \cap B} f \in M(T)$ and $1_{A \cap B^c} f \in M(T)$, and so $A \cap B \in J(T)$ and $A \cap B^c \in J(T)$. Hence A is not an atom of $J(T)$.

Remark 5. Let $T \in \mathscr{L}_+(L^p(m), L^p(n))$ be such that $\operatorname{supp} T \in J(T)$ and $J(T)$ is purely atomic. Let $A_1, A_2, \ldots$ be all atoms of $J(T)$. As easily seen, $(TI_{A_i})^*$ have disjoint supports. Hence for every $g \in M(T)$ we have $1_{A_i} g \in M(TI_{A_i})$. Fix $f \in M(T)$ with $f \geqslant 0$ and $\operatorname{supp} f = \operatorname{supp} T$. From Remark 4 and Theorem 4 it follows that $1_{A_i} g = a_i 1_{A_i} f$ for some $a_i \in R$. As, moreover, $\operatorname{supp} g \subset \operatorname{supp} f$, we infer that

$$M(T) = \Big\{\sum_i a_i 1_{A_i} f \colon \sum_i |a_i|^p \|1_{A_i} f\|_p^p < \infty\Big\}.$$

Thus $M(T)$ is isometric and order isomorphic to l^p or l^p_n according as the set of atoms of $J(T)$ is infinite or has the cardinality n.

Added in proof. For a related result see the author, *A characterization of l^p-spaces in terms of positive operators*, Bulletin of the Polish Academy of Sciences, Mathematics, 33 (1985), p. 377–379.

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