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ON A CONJECTURE OF ERDÖS

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Let A be the set of all integers N such that all numbers $N-2^k$ $(1 \leqslant k \leqslant (\log N)/\log 2)$ are primes and let $A(x)$ be the counting function of A. Erdös [1] conjectured that

$$A = \{7, 15, 21, 45, 75, 105\}.$$

Later, Vaughan [4] proved the inequality

$$A(X) \ll X \exp\left\{-C\frac{\log X \log\log\log X}{\log\log X}\right\} \tag{1}$$

using the large sieve.

The aim of this note is to point out that assuming the extended hypothesis of Riemann it is possible to obtain the bound

$$A(X) \underset{\varepsilon}{\ll} X^{\vartheta+\varepsilon}, \tag{2}$$

where

$$\vartheta = 1-\frac{\lambda}{\log 2} \quad \text{and} \quad \lambda = \prod_p \left(1-\frac{1}{p(p-1)}\right).$$

In particular, $A(X) = O(\sqrt{X})$ is implied by ERH. We start with the observation made first in [3] that if $N \in A$, then N is divisible by any prime $p \leqslant (\log N)/(\log 2)$ belonging to P_2, the set of all primes having 2 for a primitive root. Indeed, if such a p does not divide N, then with a suitable $k \leqslant p-1$ we have $0 < N-2^k \equiv 0 \pmod p$, hence

$$N = 2^k+p \leqslant 2^{p-1}+p < 2^p \leqslant N.$$

Hooley [2] has proved that ERH implies that P_2 is infinite and has a Dirichlet density equal to λ. This implies

$$\prod_{\substack{p \in P_2 \\ p \leqslant t}} p = \exp\Big\{\sum_{\substack{p \in P_2 \\ p \leqslant t}} \log p\Big\} = \exp\{(\lambda+o(1))t\} \underset{\varepsilon}{\gg} \exp\{(\lambda-\varepsilon)t\}$$

for every $\varepsilon > 0$.

Now let $M = M(X)$ be chosen with $2^M < X$ and put

$$D_M = \prod_{\substack{p \in P_2 \\ p \leqslant M}} p.$$

Then

$$A(X) \leqslant A(2^M) + \mathcal{N}\{2^M < n \leqslant X\colon D_M | n\} \leqslant \frac{X}{D_M} + A(2^M).$$

Putting $M(X) = a \log X$ with $a < 1/\log 2$ we get

$$A(X) \underset{\varepsilon}{\ll} X^{1-\lambda a+\varepsilon} + A(X^{a\log 2}).$$

Now we need a simple lemma:

If $A(X) \leqslant X$ is a non-negative function, satisfying for certain $0 < b < a$, all $\varepsilon > 0$, and all $0 < \alpha < 1/a$ the condition

$$A(X) \underset{\varepsilon}{\ll} A(X^{a\alpha}) + X^{1-b\alpha+\varepsilon}, \tag{3}$$

then, for every $\varepsilon > 0$,

$$A(X) \underset{\varepsilon}{\ll} X^{1-(b/a)+\varepsilon}. \tag{4}$$

To prove this observe first that (3) implies

$$A(X) \underset{\varepsilon}{\ll} X^{1-b\alpha+\varepsilon} + X^{a\alpha}.$$

Putting here $\alpha = 1/(a+b)$ we obtain

$$A(X) \ll 2X^{a/(a+b)+\varepsilon}.$$

Thus (3) takes the form

$$A(X) \underset{\varepsilon}{\ll} 2X^{a\alpha^2/(a+b)+\varepsilon} + X^{1-b\alpha+\varepsilon}.$$

Choosing $\alpha = (a+b)/(a^2+ab+b^2)$ we obtain

$$A(X) \underset{\varepsilon}{\ll} 3X^{[1+b/a+(b/a)^2]^{-1}+\varepsilon}.$$

By an easy induction, for every k we get the evaluation

$$A(X) \underset{\varepsilon}{\ll} kX^{1-\beta+\beta^{k+1}+\varepsilon}$$

with $\beta = b/a < 1$. Taking k sufficiently large we obtain (4).

Now it suffices to apply the lemma in the case $a = \log 2$ and $b = \lambda$.

Added in proof. The evaluation $A(X) \underset{\varepsilon}{\ll} X^{1-\lambda+\varepsilon}$ is deduced from ERH in the same way in C. Hooley, *Applications of sieve methods*, Academic Press 1974, Chapter VII.

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