COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 2

A REMARK ON v^*-ALGEBRAS

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In this note we adopt the definitions and notations given by E. Marczewski in [1]. Let $\mathfrak{A} = (A; \mathbf{F})$ be an abstract algebra. For any non-void set $E \subset A$ we denote by $C(E)$ the subalgebra generated by E, $C(\emptyset)$ denoting the set ot all algebraic constants. For finite sets $E = \{a_1, a_2, \ldots, \ldots, a_n\}$ we shall also use the notation $C(E) = C(a_1, a_2, \ldots, a_n)$. The set of independent generators of an algebra is called a *basis* of this algebra.

An algebra $(A; \mathbf{F})$ is called a *v^*-algebra* if it satisfies the following conditions:

(*) each self-dependent element of A is an algebraic constant,

(**) if the elements $a_1, a_2, \ldots, a_n$ $(n \geqslant 1)$ are independent and the elements $a_1, a_2, \ldots, a_n, a_{n+1}$ are dependent $(a_1, a_2, \ldots, a_{n+1} \in A)$, then $a_{n+1} \in C(a_1, a_2, \ldots, a_n)$.

W. Narkiewicz proved in [3] that the independence in v^*-algebras has the principal properties of linear independence. A representation theorem for v^*-algebras was proved in [5], [6] and [7]. The purpose of this note is to prove a theorem which gives an affirmative answer to a problem raised by S. Fajtlowicz (The New Scottish Book, Problem 792).

THEOREM. *If an algebra $\mathfrak{A}$ satisfies the following conditions*

(i) *each set of independent elements in $\mathfrak{A}$ can be extended to a basis of $\mathfrak{A}$,*

(ii) *each subalgebra of $\mathfrak{A}$ either consists of algebraic constants of $\mathfrak{A}$ or has a basis consisting of independent elements of $\mathfrak{A}$,*

then it is a v^*-algebra.

We note that v^*-algebras have the properties (i) and (ii) (see [3]). Thus the theorem gives a characterization of v^*-algebras. Before proving the theorem we shall prove some lemmas.

Given a subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$, we put $\gamma_{\mathfrak{A}}(\mathfrak{B}) = 0$ if all elements of the carrier of $\mathfrak{B}$ are algebraic constants in $\mathfrak{A}$. In the remaining case, if $\mathfrak{B}$ is finitely generated, then $\gamma_{\mathfrak{A}}(\mathfrak{B})$ is the minimal number of generators of $\mathfrak{B}$ and $\gamma_{\mathfrak{A}}(\mathfrak{B}) = \infty$ if $\mathfrak{B}$ is not finitely generated. Further, $\iota_{\mathfrak{A}}(\mathfrak{B}) = \infty$ if the carrier of $\mathfrak{B}$ contains sets of every finite power consisting of independent elements of $\mathfrak{A}$. In the remaining case $\iota_{\mathfrak{A}}(\mathfrak{B})$ is defined as the maximal number of elements belonging to the carrier of $\mathfrak{B}$ and independent in $\mathfrak{A}$. The constants $\gamma_{\mathfrak{A}}(\mathfrak{A})$ and $\iota_{\mathfrak{A}}(\mathfrak{A})$ denoted by $\gamma(\mathfrak{A})$ and $\iota(\mathfrak{A})$ respectively were introduced and investigated by E. Marczewski in [2].

In what follows we shall consider the algebra $\mathfrak{A}$ with the properties (i) and (ii).

LEMMA 1. *No subalgebra of $\mathfrak{A}$ containing an infinite set of elements independent in $\mathfrak{A}$ is finitely generated.*

Proof. Suppose the contrary. Let $b_1, b_2, \ldots$ be a sequence of independent elements of $\mathfrak{A}$ belonging to a finitely generated subalgebra $C(a_1, a_2, \ldots, a_n)$. By (i) the set $b_1, b_2, \ldots$ can be extended to a basis of $\mathfrak{A}$. Consequently, there exist elements $c_1, c_2, \ldots, c_m$ and $(k+m)$-ary algebraic operations $f_1, f_2, \ldots, f_n$ such that the elements $c_1, c_2, \ldots$ $\ldots, c_m, b_1, b_2, \ldots$ are independent in $\mathfrak{A}$ and

$$a_j = f_j(b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_m) \quad (j = 1, 2, \ldots, n).$$

Hence it follows that the elements $c_1, c_2, \ldots, c_m, b_1, b_2, \ldots$ belong to the subalgebra $C(b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_m)$ which contradicts the independence of $c_1, c_2, \ldots, c_m, b_1, b_2, \ldots$ The lemma is thus proved.

By Theorem 3 in [4] if $\mathfrak{A}$ has n generators and $n+1$ independent elements, then it contains an infinite set of independent elements. Consequently, as a consequence of Lemma 1 we have the following

COROLLARY 1. $\iota(\mathfrak{A}) \leqslant \gamma(\mathfrak{A})$.

LEMMA 2. *For each subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ the formula $\iota_{\mathfrak{A}}(\mathfrak{B}) = \gamma_{\mathfrak{A}}(\mathfrak{B})$ holds.*

Proof. By (ii) for each subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ we have the inequality $\iota_{\mathfrak{A}}(\mathfrak{B}) \geqslant \gamma_{\mathfrak{A}}(\mathfrak{B})$. Suppose that the assertion of the Lemma is not true. There exists then a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ for which the inequality $\iota_{\mathfrak{A}}(\mathfrak{C}) > \gamma_{\mathfrak{A}}(\mathfrak{C})$ holds. Hence we get inequality

$$(1) \qquad 1 \leqslant \gamma_{\mathfrak{A}}(\mathfrak{C}) < \infty.$$

Moreover, by Corollary 1, $\mathfrak{C} \neq \mathfrak{A}$. Put $k = \gamma_{\mathfrak{A}}(\mathfrak{C})$. Let $a_1, a_2, \ldots, a_k$ be generators of $\mathfrak{C}$ and $b_1, b_2, \ldots, b_{k+1}$ independent in $\mathfrak{A}$ elements of $\mathfrak{C}$. Evidently, for some algebraic k-ary operations $f_1, f_2, \ldots, f_{k+1}$ we have

$$(2) \qquad b_j = f_j(a_1, a_2, \ldots, a_k) \quad (j = 1, 2, \ldots, k+1).$$

We define by induction the $(n+k)$-tuples $c_1, c_2, \ldots, c_n, u_1^{(n)}, u_2^{(n)}, \ldots, \ldots, u_k^{(n)}$ $(n = 1, 2, \ldots)$ of elements of $\mathfrak{C}$ setting

$$c_1 = b_1, \quad u_j^{(1)} = b_{j+1} \quad (j = 1, 2, \ldots, k),$$

$$(3) \qquad c_{n+1} = f_1(u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}),$$

$$(4) \qquad u_j^{(n+1)} = f_{j+1}(u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}) \quad (j = 1, 2, \ldots, k).$$

We assert that for all n the elements $c_1, c_2, \ldots, c_n, u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}$ are independent in $\mathfrak{A}$. Indeed, this is true for $n = 1$. Suppose that $c_1, c_2, \ldots, c_n, u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}$ are independent in $\mathfrak{A}$. To show the independence of $c_1, c_2, \ldots, c_{n+1}, u_1^{(n+1)}, u_2^{(n+1)}, \ldots, u_k^{(n+1)}$ we ought to prove that if, for some algebraic operations f and g, the equation

$$(5)\qquad f(c_1, c_2, \ldots, c_{n+1}, u_1^{(n+1)}, u_2^{(n+1)}, \ldots, u_k^{(n+1)}) \\ = g(c_1, c_2, \ldots, c_{n-1}, u_1^{(n+1)}, u_2^{(n+1)}, \ldots, u_k^{(n+1)})$$

holds, then $f = g$ identically in the algebra $\mathfrak{A}$. Since $\mathfrak{C} \neq \mathfrak{A}$, the set $\{c_1, c_2, \ldots, c_n, u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}\}$ is not a basis of the algebra $\mathfrak{A}$. On the other hand, by (i), it can be extended to a basis of $\mathfrak{A}$. Consequently, by Corollary 1, we have the inequality $\gamma(\mathfrak{A}) \geqslant n+k+1$. Thus, applying (i) to the set $\{b_1, b_2, \ldots, b_{k+1}\}$ of independent elements, we infer that there exist elements $d_1, d_2, \ldots, d_n$ in the algebra $\mathfrak{A}$ such that $d_1, d_2, \ldots$ $\ldots, d_n, b_1, b_2, \ldots, b_{k+1}$ are independent. From (3), (4) and (5) we obtain the equation

$$(6)\qquad f\big(c_1, c_2, \ldots, c_n, f_1(u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}), \ldots, f_{k+1}(u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)})\big) \\ = g\big(c_1, c_2, \ldots, c_n, f_1(u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}), \ldots, f_{k+1}(u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)})\big).$$

By the independence of $c_1, c_2, \ldots, c_n, u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}$ it follows that (6) will be preserved if we substitute for $c_1, c_2, \ldots, c_n$ the elements $d_1, d_2, \ldots, d_n$ and for $u_1^{(n)}, u_2^{(n)}, \ldots, u_k^{(n)}$ the elements $a_1, a_2, \ldots, a_k$. Then, by (2), equation (6) passes into equation

$$f(d_1, d_2, \ldots, d_n, b_1, b_2, \ldots, b_{k+1}) = g(d_1, d_2, \ldots, d_n, b_1, b_2, \ldots, b_{k+1})$$

which, by the independence of $d_1, d_2, \ldots, d_n, b_1, b_2, \ldots, b_{k+1}$ implies the identity $f = g$. Thus, for every n, the elements $c_1, c_2, \ldots, c_n, u_1^{(n)}, u_2^{(n)}, \ldots$ $\ldots, u_k^{(n)}$ are independent in the algebra $\mathfrak{A}$. Hence, in particular, it follows that the elements $c_1, c_2, \ldots$ from $\mathfrak{C}$ are independent in the algebra $\mathfrak{A}$. Consequently, by Lemma 1, the subalgebra $\mathfrak{C}$ is not finitely generated which contradicts (1). The Lemma is thus proved.

LEMMA 3. *If the elements $a_1, a_2, \ldots, a_n$ belong to a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with $\gamma_{\mathfrak{A}}(\mathfrak{B}) = n \geqslant 1$ and are independent in $\mathfrak{A}$, then they generate $\mathfrak{B}$.*

Proof. By Lemma 2, $\iota_{\mathfrak{A}}(\mathfrak{B}) = n$. Consequently, by (ii), the subalgebra $\mathfrak{B}$ has an n-element basis $b_1, b_2, \ldots, b_n$ consisting of elements independent in $\mathfrak{A}$. By (i) the set $\{b_1, b_2, \ldots, b_n\}$ can be extended to a basis $\{b_1, b_2, \ldots, b_n\} \cup C$ of the algebra $\mathfrak{A}$.

First we shall prove that the set $\{a_1, a_2, \ldots, a_n\} \cup C$ consists of independent elements. Let $f_1, f_2, \ldots, f_n$ be these algebraic operations for which

$$(7)\qquad a_j = f_j(b_1, b_2, \ldots, b_n) \quad (j = 1, 2, \ldots, n).$$

By (i) the set $\{a_1, a_2, \ldots, a_n\}$ of independent elements can be extended to a basis $\{a_1, a_2, \ldots, a_n\} \cup D$ of $\mathfrak{A}$. By Lemma 2 and Theorem (iv) in [1], we have the formula

$$\text{card}\, C = \text{card}\, D. \tag{8}$$

Let $c_1, c_2, \ldots, c_m \in C$ and, for some algebraic operations f and g, $f(a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_m) = g(a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_m)$. Substituting (7) into the last equation we obtain

$$\begin{aligned}(9)\quad & f\big(f_1(b_1, b_2, \ldots, b_n), \ldots, f_n(b_1, b_2, \ldots, b_n), c_1, c_2, \ldots, c_m\big) \\ & \qquad = g(f_1(b_1, b_2, \ldots, b_n), \ldots, f_n(b_1, b_2, \ldots, b_n), c_1, c_2, \ldots, c_m).\end{aligned}$$

From the independence of $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_m$ it follows that (9) will be preserved if we substitute for $c_1, c_2, \ldots, c_m$ some elements $d_1, d_2, \ldots, d_m$ from D. This substitution is possible by virtue of (8) and leads, according to (7), to the equation

$$f(a_1, a_2, \ldots, a_n, d_1, d_2, \ldots, d_m) = g(a_1, a_2, \ldots, a_n, d_1, d_2, \ldots, d_m).$$

Now, taking into account the independence of $a_1, a_2, \ldots, a_n, d_1, d_2, \ldots \ldots, d_m$, we get the identity $f = g$. Thus the set $\{a_1, a_2, \ldots, a_n\} \cup C$ consists of independent elments.

Now we shall prove that it is a basis of $\mathfrak{A}$. Suppose the contrary. By (i) there exists an element d in the algebra $\mathfrak{A}$ such that the elements of the set $\{d, a_1, a_2, \ldots, a_n\} \cup C$ are independent. Since the set $\{b_1, b_2, \ldots \ldots, b_n\} \cup C$ is a basis of $\mathfrak{A}$, we can find elements $v_1, v_2, \ldots, v_p$ in C such that $d \in C(b_1, b_2, \ldots, b_n, v_1, v_2, \ldots, v_p) = \mathfrak{C}$. Evidently, $\gamma_{\mathfrak{A}}(\mathfrak{C}) \leqslant n+p$. On the other hand, the subset $\{d, a_1, a_2, \ldots, a_n, v_1, v_2, \ldots, v_p\}$ of the carrier of $\mathfrak{C}$ consists of independent in $\mathfrak{A}$ elements. Thus $\iota_{\mathfrak{A}}(\mathfrak{C}) \geqslant n+p+1$ which contradicts Lemma 2. Thus the set $\{a_1, a_2, \ldots, a_n\} \cup C$ is a basis of $\mathfrak{A}$.

Let $g_1, g_2, \ldots, g_n$ and $w_1, w_2, \ldots, w_q$ be these algebraic operations in $\mathfrak{A}$ and elements of C respectively for which

$$b_j = g_j(a_1, a_2, \ldots, a_n, w_1, w_2, \ldots, w_q) \quad (j = 1, 2, \ldots, n).$$

Hence and from (7) it follows that

$$\begin{aligned}(10)\quad b_j = g_j\big(f_1(b_1, b_2, \ldots, b_n), \ldots, f_n(b_1, b_2, \ldots, b_n), w_1, w_2, \ldots, w_q\big) \\ (j = 1, 2, \ldots, n).\end{aligned}$$

From the independence of $b_1, b_2, \ldots, b_n, w_1, w_2, \ldots, w_q$ it follows that (10) will be preserved if we substitute for $w_1, w_2, \ldots, w_q$ the element a_1. Consequently, by (7),

$$b_j = g_j(a_1, a_2, \ldots, a_n, a_1, a_1, \ldots, a_1) \quad (j = 1, 2, \ldots, n)$$

which shows that the elements $a_1, a_2, \ldots, a_n$ generate the subalgebra $\mathfrak{B}$. The Lemma is thus proved.

LEMMA 4. *If $\gamma_{\mathfrak{A}}(\mathfrak{B}) = n \geqslant 1$ and the elements $a_1, a_2, \ldots, a_n$ generate the subalgebra $\mathfrak{B}$, then they are independent in $\mathfrak{A}$.*

Proof. By (ii) and Lemma 2 there exists an n-element basis $b_1, b_2, \ldots$ $\ldots, b_n$ of $\mathfrak{B}$. Let $f_1, f_2, \ldots, f_n$ be these algebraic operations for which

$$b_j = f_j(a_1, a_2, \ldots, a_n) \quad (j = 1, 2, \ldots, n). \tag{11}$$

Put

$$c_j = f_j(b_1, b_2, \ldots, b_n) \quad (j = 1, 2, \ldots, n). \tag{12}$$

By a Marczewski's theorem ([1], p. 60) the elements $c_1, c_2, \ldots, c_n$ are independent in $\mathfrak{A}$ and, consequently, by Lemma 3, form a basis of the subalgebra $\mathfrak{B}$. Taking into account the representation

$$b_j = g_j(c_1, c_2, \ldots, c_n) \quad (j = 1, 2, \ldots, n), \tag{13}$$

where $g_1, g_2, \ldots, g_n$ are algebraic operations, we have, by (12), the equation

$$b_j = g_j\big(f_1(b_1, b_2, \ldots, b_n), \ldots, f_n(b_1, b_2, \ldots, b_n)\big) \quad (j = 1, 2, \ldots, n). \tag{14}$$

From the independence of $b_1, b_2, \ldots, b_n$ it follows that (14) will be preserved if we substitute for $b_1, b_2, \ldots, b_n$ the elements $a_1, a_2, \ldots, a_n$. It follows from (11) that, after this substitution, (14) passes into the equation

$$a_j = g_j(b_1, b_2, \ldots, b_n) \quad (j = 1, 2, \ldots, n).$$

Hence and from (13), by virtue of a Marczewski's theorem ([1], p. 60) we get the independence of $a_1, a_2, \ldots, a_n$ which completes the proof.

Proof of the Theorem. Suppose that the algebra $\mathfrak{A}$ has properties (i) and (ii). Let a be a self-dependent element of $\mathfrak{A}$. By Lemma 4 the inequality $\gamma_{\mathfrak{A}}\big(C(a)\big) > 0$ would imply the independence of a. Thus $\gamma_{\mathfrak{A}}\big(C(a)\big) = 0$ and, consequently, a is an algebraic constant in $\mathfrak{A}$. Condition (*) is thus proved.

To prove condition (**) let us suppose that the elements $a_1, a_2, \ldots, a_n$ are independent in $\mathfrak{A}$ and the elements $a_1, a_2, \ldots, a_n, a_{n+1}$ are dependent in $\mathfrak{A}$. By Lemmas 2 and 4 we have the formula $\gamma_{\mathfrak{A}}\big(C(a_1, a_2, \ldots, a_{n+1})\big) = n$. Hence and from Lemma 3 it follows that the elements $a_1, a_2, \ldots, a_n$ generate the subalgebra $C(a_1, a_2, \ldots, a_{n+1})$ which implies the condition (**). Thus $\mathfrak{A}$ is a v^*-algebra.

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Reçu par la Rédaction le 4. 5. 1968