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EXTENSIONS OF POSITIVE OPERATORS AND EXTREME POINTS. II

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The present paper is a continuation of author's work [4], joint with Plachky and Thomsen. Nevertheless, its first section can be read independently of [4] and the other two depend only on Theorem 3 from [4]. The latter result is repeated at the beginning of Section 2.

In Section 1 we improve upon a classical theorem of L. V. Kantorovič concerning extension of positive operators on ordered vector spaces. Our main result (Theorem 1) shows the existence of extreme points in the set of all positive extensions of a given operator with values in an order complete vector lattice.

Section 2 deals with extension of lattice homomorphisms. It is proved that an extreme positive extension of a lattice homomorphism is again a lattice homomorphism and a converse to this assertion also holds (Theorem 2).

Finally, in Section 3, applying Theorem 3 of [4] and Corollary 3 of Section 2, we give a characterization of the extreme points of the set of all positive operators between two vector lattices of measurable functions taking 1 into 1. Our characterization is akin to some results of Phelps [6] and Iwanik [2].

Throughout the paper we adhere to the terminology of Schaefer's monograph [7]. We use the following notation which coincides with that of [4]. X stands for an ordered real vector space and M for its vector subspace. Y stands for an order complete real vector lattice. Given $T \in L_+(M, Y)$ (i.e. a positive linear operator from M into Y) and a vector subspace N of X with $M \subset N$, we put

$$E(T, N) = \{S \in L_+(N, Y) \colon S|M = T\}.$$

The notation $E(T, X)$ is abbreviated to $E(T)$.

The author is indebted to Doctors A. Iwanik and W. Thomsen for stimulating remarks on the subject of this paper.

1. Extreme positive extensions. As in [4], we associate with every $T \in L_+(M, Y)$ a map $T_i: X \to Y \cup \{\pm\infty\}$ defined by

$$T_i(x) = \sup\{T(z): x \geqslant z \in M\} \quad \text{for all } x \in X.$$

We shall need two lemmas.

LEMMA 1 (cf. [4], (i)). *If $T \in L_+(M, Y)$, $x_0 \in X$ and $T_i(x_0) \in Y$, then*

$$\operatorname{extr} E\big(T, \operatorname{lin}(M \cup \{x_0\})\big) \neq \emptyset.$$

Proof. For $z \in M$ and $t \in R$ we put $S(z+tx_0) = T(z)+tT_i(x_0)$. Then $S \in E\big(T, \operatorname{lin}(M \cup \{x_0\})\big)$. Suppose that $S = tS'+(1-t)S''$, where $S', S'' \in E\big(T, \operatorname{lin}(M \cup \{x_0\})\big)$ and $0 < t < 1$. Then $S'(x_0)$, $S''(x_0) \geqslant S(x_0)$, so that $S'(x_0) = S''(x_0) = S(x_0)$. Hence $S' = S'' = S$.

LEMMA 2. *Suppose that $T \in L_+(M, Y)$ and N, N_1 are subspaces of X with $M \subset N \subset N_1$. If $S \in \operatorname{extr} E(T, N)$ and $S_1 \in \operatorname{extr} E(S, N_1)$, then $S_1 \in \operatorname{extr} E(T, N_1)$.*

Proof. Suppose that $S_1 = tS'+(1-t)S''$, where $S', S'' \in E(T, N_1)$ and $0 < t < 1$. Then $S'|N$, $S''|N \in E(T, N)$, whence $S'|N = S''|N = S$. Thus $S', S'' \in E(S, N_1)$, and so $S' = S'' = S_1$.

THEOREM 1 (cf. [4], Theorem 1, and [3], 1.8.4). *If $T \in L_+(M, Y)$, where M is a majorizing subspace of X, then* $\operatorname{extr} E(T) \neq \emptyset$.

Proof. Consider the class $\boldsymbol{M}$ of all pairs (N, S), where N is a subspace of X, $M \subset N$ and $S \in \operatorname{extr} E(T, N)$. If (N_1, S_1), $(N_2, S_2) \in \boldsymbol{M}$, we write $(N_1, S_1) \preceq (N_2, S_2)$ provided $N_1 \subset N_2$ and $S_2 \in E(S_1, N_2)$. Clearly, $(\boldsymbol{M}, \preceq)$ is a non-empty ordered class. By assumption and Lemmas 1 and 2, for any maximal element (N, S) of $\boldsymbol{M}$ we have $N = X$. Hence, in view of the Kuratowski-Zorn lemma, it is enough to show that each chain $\{(N_a, S_a)\}$ in $\boldsymbol{M}$ is bounded. Put

$$N_0 = \bigcup_a N_a \quad \text{and} \quad S_0(x) = S_a(x) \text{ if } x \in N_a.$$

Then, as easily seen, $(N_0, S_0) \in \boldsymbol{M}$ and $(N_a, S_a) \preceq (N_0, S_0)$ for all a.

COROLLARY 1 (cf. [4], (ii)). *Suppose that X has an order unit u. If $T \in L_+(M, Y)$ and $T_i(u) \in Y$, then* $\operatorname{extr} E(T) \neq \emptyset$.

Proof. By Lemma 1, $\operatorname{extr} E\big(T, \operatorname{lin}(M \cup \{u\})\big) \neq \emptyset$. As $\operatorname{lin}(M \cup \{u\})$ majorizes X, the assertion follows from Theorem 1 and Lemma 2.

2. Extension of lattice homomorphisms. Throughout the rest of the paper X is assumed to be a vector lattice. Under this assumption, it is known that $S \in \operatorname{extr} E(T)$ if and only if $S \in E(T)$ and $\inf\{S(|x-z|): z \in M\} = 0$ for each $x \in X$ ([4], Theorem 3).

We denote by $H(X, Y)$ the set of all lattice homomorphisms of X into Y ([7], Definition II.2.4). Recall that $T \in H(X, Y)$ if and only if $T \in L(X, Y)$ and $|T(x)| = T(|x|)$ for each $x \in X$ ([7], Proposition II.2.5).

THEOREM 2. *Let M be a vector sublattice of X and let $T \in H(M, Y)$. Then*

(a) $\operatorname{extr} E(T) \subset H(X, Y)$.

(b) *If* $\inf\{|y - T(z)|: z \in M\} = 0$ *for each* $y \in Y$, *then*

$$E(T) \cap H(X, Y) \subset \operatorname{extr} E(T).$$

Proof. (a) Suppose that $S \in \operatorname{extr} E(T)$. As S is positive, it is enough to show that $S(|x|) \leqslant |S(x)|$ for each $x \in X$. We have

$$S(|x|) \leqslant S(|z|) + S(|x - z|) = |S(z)| + S(|x - z|) \leqslant |S(x)| + 2S(|x - z|)$$

provided $z \in M$. Hence the assertion follows from Theorem 3 of [4] (see above).

(b) Suppose that $S \in E(T) \cap H(X, Y)$. Then

$$S(|x - z|) = |S(x - z)| = |S(x) - T(z)|,$$

so that, on account of our assumption, the assertion follows by another application of Theorem 3 of [4].

Note that the assumption in (b) cannot be dropped. Indeed, take $X = Y$ and $M = \{0\}$. Then the identity map I on X is not an extreme extension of $I|M$ unless $X = \{0\}$.

COROLLARY 2. *Let M be a majorizing vector sublattice of X. Then any lattice homomorphism $T: M \to Y$ extends to a lattice homomorphism $S: X \to Y$. In case $M/T^{-1}(0)$ is order complete, S can be chosen in such a way that for each $x \in X$ there is $z \in M$ with $S(x) = T(z)$.*

Proof. The first part follows immediately from Theorems 1 and 2 (a). Applying it, in case $M/T^{-1}(0)$ is order complete, to the canonical lattice homomorphism of M onto $M/T^{-1}(0)$, we get a lattice homomorphism $q: X \to M/T^{-1}(0)$. Let $\tilde{T}: M/T^{-1}(0) \to Y$ be the quotient map of T and put $S = \tilde{T}q$. Clearly, $S \in H(X, Y) \cap E(T)$. Moreover, given $x \in X$, there is $z \in M$ with $q(x) = q(z)$, so that $S(x) = T(z)$.

In case $Y = R$ and v is an order unit in X, the next result is due to Bonsall ([1], the Corollary to Theorem 1). That this is so can be seen using a well-known characterization of lattice homomorphisms ([3], 2.6.7, or [7], the Corollary to Proposition II.2.6).

COROLLARY 3. *Let $w \in Y_+$ and $\inf\{|y - tw|: t \in R\} = 0$ for each $y \in Y$. Suppose that $v \in X_+$, $S \in L_+(X, Y)$ and $S(v) = w$. Then $S \in \operatorname{extr}\{T \in L_+(X, Y): T(v) = w\}$ if and only if $S \in H(X, Y)$.*

Proof. Put $M = \{tv: t \in R\}$ and $T_0(tv) = tw$ for $t \in R$. Then

$$T_0 \in H(M, Y) \quad \text{and} \quad E(T_0) = \{T \in L_+(X, Y): T(v) = w\}.$$

Hence an application of Theorem 2 yields the result.

Remark. Under different assumptions on X, Y, w and v the equivalence given in Corollary 3 has been obtained by R. J. Nagel ([7], Proposition III.9.2).

3. Extreme positive operators between vector lattices of measurable functions. Let $(\Omega_i, \Sigma_i, \mu_i)$, where $i = 1, 2$, be positive finite measure spaces. Denote by $L_0(\mu_i)$ the (order complete) vector lattice of (μ_i-equivalence classes of) real-valued measurable functions on Ω_i and by $s(\mu_i)$ its vector sublattice consisting of all simple functions.

We shall need the following

LEMMA 3. *Suppose $S \in L_+(s(\mu_1), L_0(\mu_2))$, $S1_{\Omega_1} = 1_{\Omega_2}$ and $A \in \Sigma_1$. Then $S1_A$ is a characteristic function if and only if*

$$\inf\{S(|1_A - t1_{\Omega_1}|):\ t \in R\} = 0.$$

Proof. As $S(|1_A - t1_{\Omega_1}|) = |1-t|S1_A + |t|S1_{A^c}$, we have

$$\inf\{S(|1_A - t1_{\Omega_1}|):\ t \in R\} = (S1_A) \wedge (S1_{A^c}).$$

By assumption, $S1_A + S1_{A^c} = 1_{\Omega_2}$. Hence $S1_A$ is a characteristic function if and only if $(S1_A) \wedge (S1_{A^c}) = 0$. From these statements we get the assertion.

The next theorem generalizes Propositions I.4.3 and 4 in [7] on stochastic matrices. It is also akin to some results of Phelps ([6], Theorem 2.2) and Iwanik ([2], Lemma 2 and Proposition 2).

THEOREM 3. *Let X be a vector sublattice of $L_0(\mu_1)$ containing $s(\mu_1)$ and let Y be an order complete vector sublattice of $L_0(\mu_2)$. Suppose*

(*) *given $x \in X$, there exist $x_n \in s(\mu_1)$, $v \in X_+$ and $\varepsilon_n \in R_+$ with $|x - x_n| \leqslant \varepsilon_n v$ and $\varepsilon_n \downarrow 0$.*

Then for each $S \in L_+(X, Y)$ with $S1_{\Omega_1} = 1_{\Omega_2}$ the following three conditions are equivalent:

(i) $S \in \operatorname{extr}\{T \in L_+(X, Y):\ T1_{\Omega_1} = 1_{\Omega_2}\}$.

(ii) *S takes characteristic functions into characteristic functions.*

(iii) $S \in H(X, Y)$.

Proof. The equivalence (i) $\Leftrightarrow$ (iii) follows readily from Corollary 3. We shall show that (i) is also equivalent to (ii). Our assumptions on X and Y allow us to apply Theorem 3 of [4] with $M = \{t1_{\Omega_1}:\ t \in R\}$. Accordingly, we see that (i) holds if and only if $\inf\{S(|x - t1_{\Omega_1}|):\ t \in R\} = 0$ for each $x \in X$. Replacing "X" by "$s(\mu_1)$" in the latter condition, we obtain a further equivalent condition. This follows from (*) as Y is Archimedean. Finally, the set of all $x \in X$ such that $\inf\{S(|x - t1_{\Omega_1}|):\ t \in R\} = 0$ being a vector subspace of X, the desired equivalence is a consequence of Lemma 3.

Remarks. 1. In view of a result of Peressini ([5], Chapter 4, Proposition 2.4), the assumption $(*)$ is satisfied in case X is a topological vector lattice the topology of which is complete and metrizable, and $s(\mu_1)$ is a topologically dense subset of X. In particular, Theorem 3 holds with $X = L_{p_1}(\mu_1)$ and $Y = L_{p_2}(\mu_2)$, where $0 \leqslant p_1, p_2 \leqslant \infty$.

2. The equivalence of conditions (ii) and (iii) in Theorem 3 can also be proved directly using the arguments given in [2], p. 175.

3. The role played by the measure μ_i, $i = 1, 2$, in Theorem 3 is limited to generating the ideal $\mathcal{N}_i$ of null sets in Σ_i. We leave it to the reader to establish which properties of $\mathcal{N}_i$ are essential in the proof.

4. Under the assumptions of Theorem 3 conditions (i)-(iii) are equivalent to

(iv) $S(1_A \cdot 1_B) = (S1_A)\cdot(S1_B)$ *for all* $A, B \in \Sigma_1$.

Indeed, (iv) $\Rightarrow$ (ii) and (ii) $\wedge$ (iii) $\Rightarrow$ (iv). Using $(*)$, one can also verify that if X and Y are additionally closed under multiplication, then (iv) is equivalent to

(iv)′ S *is multiplicative* (cf. [6]).

Added in proof. 1. A different proof of the first part of Corollary 2 can be found in another paper by the author (*Extension of vector-lattice homomorphisms*, Proceedings of the American Mathematical Society, in print).

2. Dr. W. Thomsen has recently proved that the approximation condition which appears in Corollary 3 is equivalent to w being a weak order unit.

3. Part III of the present paper has been recently accepted for publication in this journal.

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