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ON SOME CONDITIONS FOR THE ALMOST EVERYWHERE CONVERGENCE OF A CLASS OF INTEGRABLE FUNCTIONS

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Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space and let $\{f_k;\ k \geqslant 0\}$ be a sequence of integrable functions on Ω. Duncan(1) investigated the almost everywhere (a.e.) convergence of the sequence $\{f_k\}$ under some additional assumptions which are frequently met in practice. In this paper we extend some of these results and apply them to studying the strong law of large numbers.

For a sequence $\{f_k;\ k \geqslant 0\}$ of integrable functions on Ω and $\alpha > 0$, we put

$$B_{n,n} = B_{n,n}(\alpha) = [|f_n| > \alpha] = \{\omega\colon\ |f_n(\omega)| > \alpha\}$$

and

$$B_{n,k} = \bigcap_{j=n}^{k-1} B_{jj}^c \cap B_{kk} \quad \text{for } k > n \geqslant 0.$$

For n fixed the sets $B_{n,k}$ $(k \geqslant n)$ are disjoint. We put

$$B_n = B_n(\alpha) = \bigcup_{k \geqslant n} B_{n,k} = [\sup_{k \geqslant n} |f_k| > \alpha]$$

and $B = B(\alpha) = \bigcap_n B_n(\alpha)$. Then

$$[\overline{\lim_{k\to\infty}}\, |f_k| > \alpha] \leqslant B(\alpha) \leqslant [\overline{\lim_{k\to\infty}}\, |f_k| \geqslant \alpha].$$

It follows that $\mu[\overline{\lim}|f_k| > 0] = 0$ iff $\mu[B(\alpha)] = 0$ for each $\alpha > 0$. With this notation we have

THEOREM 1. *Suppose*

(1) $$\int |f_n| \to 0 \quad \textit{as } n \to \infty.$$

(1) R. D. Duncan, *Almost everywhere convergence of a class of integrable functions*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 31 (1975), p. 89-94.

Then, for every $\alpha > 0$,

(2) $$\overline{\lim_{n\to\infty}} \Big|\alpha\mu(B(\alpha)) - \sum_{k=n+1}^{\infty} \int_{B_{n,k}} |f_{k-1}|\Big| \leqslant 2\,\overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \int_{B_{n,k}} \frac{|f_k|^r}{|f_k|^r+\alpha^r}|f_k - f_{k-1}|$$

for all $r \geqslant 0$, *and also, for any given* $\varepsilon > 0$ *and every* $\alpha > 0$,

(3) $$\overline{\lim_{n\to\infty}} \Big|\alpha\mu(B(\alpha)) - \sum_{k=n+1}^{\infty} \int_{B_{n,k}} |f_{k-1}|\Big| < \varepsilon + 2\,\overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \int_{B_{n,k}} \frac{|f_k - f_{k-1}|^{r+1}}{|f_k - f_{k-1}|^r+\varepsilon^r}$$

for all $r \geqslant 0$.

Proof. Since $|f_k| > \alpha$ on $B_{n,k}(\alpha)$, we have

$$\begin{aligned}\alpha\mu(B_n(\alpha)) &= \alpha \sum_{k=n}^{\infty} \mu(B_{n,k}) \leqslant \sum_{k=n}^{\infty} \int_{B_{n,k}} |f_k| \\ &\leqslant \sum_{k=n}^{\infty} \int_{B_{n,k}} |f_k - f_{k-1}| + \sum_{k=n}^{\infty} \int_{B_{n,k}} |f_{k-1}| \\ &= \sum_{k=n}^{\infty} \int_{B_{n,k}} |f_k - f_{k-1}| + \sum_{k=n+1}^{\infty} \int_{B_{n,k}} |f_{k-1}| + \int_{B_{n,n}} |f_{n-1}|.\end{aligned}$$

Now, $|f_{k-1}| \leqslant \alpha$ on $B_{n,k}$ for $k \geqslant n+1$; hence

$$0 \leqslant \alpha\mu(B_n(\alpha)) - \sum_{k=n+1}^{\infty} \int_{B_{n,k}} |f_{k-1}| \leqslant \sum_{k=n}^{\infty} \int_{B_{n,k}} |f_k - f_{k-1}| + \int_{B_{n,n}} |f_{n-1}|.$$

Using the fact that $\mu(B_n(\alpha)) \downarrow \mu(B(\alpha))$, (1), and the inequality

$$\int_{B_{n,k}} |f_k - f_{k-1}| \leqslant 2 \int_{B_{n,k}} \frac{|f_k|^r}{|f_k|^r+\alpha^r}|f_k - f_{k-1}|,$$

we get (2).

To prove (3) it is enough to notice that, for any given $\varepsilon > 0$ and all $r \geqslant 0$,

$$\sum_{k=n}^{\infty} \int_{B_{n,k}} |f_k - f_{k-1}| \leqslant \varepsilon + 2 \sum_{k=n}^{\infty} \int_{B_{n,k}} \frac{|f_k - f_{k-1}|^{r+1}}{|f_k - f_{k-1}|^r+\varepsilon^r}.$$

Remark 1. Of course, if

$$|f_k - f_{k-1}| \leqslant g \in L^1(\mu) \quad \text{and} \quad |f_k - f_{k-1}| \to 0 \text{ a.e.},$$

then it follows from Theorem 1 that

$$\alpha\mu(B(\alpha)) = \lim_{n\to\infty} \sum_{k=n}^{\infty} \int_{B_{n,k}} |f_{k-1}|.$$

This is the case, e.g., if

$$f_k = S_k f - Pf,$$

where

$$S_k f = \frac{1}{k} \sum_{j=0}^{k-1} T^j f$$

with T a contraction on some $L^p(\mu)$-space, $1 < p \leqslant \infty$, and $f \in L^p(\mu)$. Here Pf is the $L^p(\mu)$-limit of $S_k f$ (op. cit.).

Consider now a sequence $\{X_k;\ k \geqslant 1\}$ of random variables on the probability space $(\Omega, \mathscr{A}, P)$. Write

$$S_k = \sum_{j=1}^{k} X_j, \qquad k \geqslant 1.$$

In this case we have

COROLLARY 1. *Suppose*

$$\text{(1')} \qquad \mathrm{E}\frac{|S_n|}{n} \to 0 \quad \text{as } n \to \infty.$$

Then, for every $\alpha > 0$,

$$\text{(2')} \qquad \overline{\lim_{n\to\infty}} \left| \alpha P(B(\alpha)) - \sum_{k=n+1}^{\infty} \mathrm{E}\left|\frac{S_{k-1}}{k-1}\right| I_{B_{n,k}} \right|$$
$$\leqslant 2 \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k(k-1)} \mathrm{E}\left(\frac{|S_k|^r}{|S_k|^r + k^r \alpha^r} |(k-1)X_k - S_{k-1}| I_{B_{n,k}} \right)$$

for all $r \geqslant 0$, *where* I_A *denotes the indicator function of* A, *and also, for any given* $\varepsilon > 0$ *and every* $\alpha > 0$,

$$\text{(3')} \qquad \overline{\lim_{n\to\infty}} \left| \alpha P(B(\alpha)) - \sum_{k=n+1}^{\infty} \mathrm{E}\left|\frac{S_{k-1}}{k-1}\right| I_{B_{n,k}} \right|$$
$$\leqslant \varepsilon + 2 \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k(k-1)} \mathrm{E}\left(\frac{|(k-1)X_k - S_{k-1}|^{r+1}}{|(k-1)X_k - S_{k-1}|^r + k^r (k-1)^r \varepsilon^r} I_{B_{n,k}} \right)$$

for all $r \geqslant 0$.

Hence, if the term on the right-hand side of (2') or (3') is zero, then the strong law of large numbers holds if and only if

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} \mathrm{E}\frac{|S_{k-1}|}{k-1} I_{B_{n,k}}(\alpha) = 0 \quad \text{for each } \alpha > 0.$$

The following lemma allows us to give some useful estimates of the right-hand sides of (2') and (3').

LEMMA. *Let $\{X_k\}$ be a sequence of integrable random variables. Then, for each $\alpha > 0$ and any given $\varepsilon > 0$,*

$$(4)\quad \frac{1}{k(k-1)}\,\mathrm{E}\left(\frac{|S_k|}{|S_k|+k\alpha}|(k-1)X_k-S_{k-1}|\,I_{B_{n,k}}\right)$$
$$\leqslant \frac{5}{2}\varepsilon\,\mathrm{E}\,I_{B_{n,k}}+2\alpha\,\mathrm{E}\,\frac{X_k^2}{k^2\varepsilon^2+X_k^2}I_{B_{n,k}}+8\alpha\,\mathrm{E}\,\frac{X_k^2}{k^2\alpha\varepsilon+|X_k|\,|S_{k-1}|}I_{B_{n,k}},$$

where $k \geqslant n$, and also

$$(5)\quad \frac{1}{k(k-1)}\,\mathrm{E}\,\frac{((k-1)X_k-S_{k-1})^2}{|(k-1)X_k-S_{k-1}|+k(k-1)\varepsilon}I_{B_{n,k}}$$
$$\leqslant 5\varepsilon\,\mathrm{E}\,I_{B_{n,k}}+6\varepsilon\,\mathrm{E}\,\frac{X_k^2}{k^2\varepsilon^2+X_k^2}I_{B_{n,k}}+12\alpha\,\mathrm{E}\,\frac{X_k^2}{k^2\alpha\varepsilon+|X_k|\,|S_{k-1}|}I_{B_{n,k}}.$$

Proof. Let us put

$$D_k = [|X_k| < k\varepsilon],\qquad E_k = [(k-1)|X_k|-|S_{k-1}| \geqslant 0],$$
$$F_k = [|X_k| > |S_{k-1}|],\qquad k = 0, 1, \ldots, X_0 = 0.$$

For any given $\varepsilon < \alpha \leqslant 1$, assuming that

$$|S_{k-1}| \leqslant \alpha(k-1)\quad\text{and}\quad |S_k| > k\alpha\quad\text{on } B_{n,k},$$

we have

$$\frac{1}{k(k-1)}\mathrm{E}\left(\frac{|S_k|}{|S_k|+k\alpha}|(k-1)X_k-S_{k-1}|\,I_{B_{n,k}}\right)$$
$$\leqslant \frac{1}{k(k-1)}\,\mathrm{E}\,\frac{|X_k|+|S_{k-1}|}{2k\alpha}\big((k-1)|X_k|+|S_{k-1}|\big)I_{B_{n,k}\cap D_k}+$$
$$+\frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)X_k^2+k|X_k|\,|S_{k-1}|+S_{k-1}^2}{|S_{k-1}+X_k|+k\alpha}I_{B_{n,k}\cap\bar{D}_k}$$
$$\leqslant \frac{\varepsilon^2}{2\alpha}\,\mathrm{E}\,I_{B_{n,k}\cap D_k}+\frac{\varepsilon}{2}\,\mathrm{E}\,I_{B_{n,k}\cap D_k}+\frac{\alpha}{2k}\,\mathrm{E}\,I_{B_{n,k}\cap D_k}+$$
$$+\frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)X_k^2+k|X_k|\,|S_{k-1}|+S_{k-1}^2}{|X_k|-|S_{k-1}|+k\alpha}I_{B_{n,k}\cap\bar{D}_k\cap F_k}+$$
$$+\frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)X_k^2+k|X_k|\,|S_{k-1}|+S_{k-1}^2}{|S_{k-1}|-|X_k|+k\alpha}I_{B_{n,k}\cap\bar{D}_k\cap\bar{F}_k}$$
$$\leqslant\left(\varepsilon+\frac{\alpha}{2k}\right)\mathrm{E}\,I_{B_{n,k}\cap D_k}+\frac{2}{k}\,\mathrm{E}\,|X_k|\,I_{B_{n,k}\cap\bar{D}_k\cap F_k}+$$

$$+\frac{\alpha}{k}\,\mathrm{E}\,I_{B_{n,k}\bar{D}_k\cap F_k}+2\alpha\mathrm{E}I_{B_{n,k}\cap\bar{D}_k\cap F_k}+\frac{\alpha}{k}\,\mathrm{E}\,I_{B_{n,k}\cap\bar{D}_k\cap\bar{F}_k}$$

$$\leqslant\left(\varepsilon+\frac{3}{2k}\right)I_{B_{n,k}}+8\alpha\,\mathrm{E}\,\frac{|X_k|}{k\alpha+|S_{k-1}|}\,\frac{|X_k|}{|X_k|+k\varepsilon}\,I_{B_{n,k}\cap\bar{D}_k\cap F_k}+$$

$$+2\alpha\,\mathrm{E}\,\frac{X_k^2}{k^2\varepsilon^2+X_k^2}\,I_{B_{n,k}\cap\bar{D}_k\cap\bar{F}_k}$$

$$\leqslant\left(\varepsilon+\frac{3}{2k}\right)I_{B_{n,k}}+8\alpha\mathrm{E}\frac{X_k^2}{k^2\alpha\varepsilon+|X_k|\,|S_{k-1}|}\,I_{B_{n,k}}+2\alpha\mathrm{E}\frac{X_k^2}{k^2\varepsilon^2+X_k^2}\,I_{B_{n,k}},$$

which proves (4) with $\alpha/k<\varepsilon$.

In an analogous way we get

$$(6)\quad \frac{1}{k(k-1)}\,\mathrm{E}\,\frac{|(k-1)\,X_k-S_{k-1}|^2}{|(k-1)\,X_k-S_{k-1}|+k(k-1)\,\varepsilon}\,I_{B_{n,k}}$$

$$\leqslant\frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)^2\,X_k^2-2(k-1)\,X_k\,|S_{k-1}|+S_{k-1}^2}{\big|(k-1)\,|X_k|-|S_{k-1}|\big|+k(k-1)\,\varepsilon}\,I_{B_{n,k}\cap D_k}+$$

$$+\frac{1}{k(k-1)}\mathrm{E}\frac{(k-1)^2\,X_k^2-2(k-1)\,X_k\,|S_{k-1}|+S_{k-1}^2}{(k-1)\,|X_k|-|S_{k-1}|+k(k-1)\,\varepsilon}\,I_{B_{n,k}\cap\bar{D}_k}.$$

But

$$(7)\quad \frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)^2\,X_k^2-2(k-1)\,X_k\,S_{k-1}+S_{k-1}^2}{\big|(k-1)\,|X_k|-|S_{k-1}|\big|+k(k-1)\,\varepsilon}\,I_{B_{n,k}\cap D_k}$$

$$\leqslant\frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)^2\,X_k^2-2(k-1)\,X_k\,S_{k-1}+S_{k-1}^2}{(k-1)\,|X_k|-|S_{k-1}|+k(k-1)\,\varepsilon}\,I_{B_{n,k}\cap D_k\cap E_k}+$$

$$+\frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)^2\,X_k^2-2(k-1)\,X_k\,S_{k-1}+S_{k-1}^2}{|S_{k-1}|-(k-1)\,|X_k|+k(k-1)\,\varepsilon}\,I_{B_{n,k}\cap D_k\cap\bar{E}_k}$$

$$\leqslant\frac{3}{\varepsilon k^2}\,\mathrm{E}\,X_k^2\,I_{B_{n,k}\cap D_k\cap E_k}+\frac{\alpha^2}{\varepsilon k^2}\,\mathrm{E}\,I_{B_{n,k}\cap D_k\cap E_k}+\frac{4\alpha^2}{k^2\,\varepsilon}\,\mathrm{E}\,I_{B_{n,k}\cap D_k\cap\bar{E}_k}$$

$$\leqslant 6\varepsilon\,\mathrm{E}\,\frac{X_k^2}{k^2\,\varepsilon^2+X_k^2}\,I_{B_{n,k}\cap D_k\cap E_k}+\frac{\alpha^2}{\varepsilon k^2}\,\mathrm{E}\,I_{B_{n,k}\cap D_k\cap E_k}+\frac{4\alpha^2}{\varepsilon k^2}\,\mathrm{E}\,I_{B_{n,k}\cap D_k\cap\bar{E}_k}$$

and

$$(8)\quad \frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)^2\,X_k^2-2(k-1)\,X_k\,S_{k-1}+S_{k-1}^2}{\big|(k-1)\,|X_k|-|S_{k-1}|\big|+(k-1)\,k\varepsilon}\,I_{B_{n,k}\cap\bar{D}_k}$$

$$=\frac{1}{k(k-1)}\,\mathrm{E}\,\frac{(k-1)^2\,X_k^2-2(k-1)\,X_k\,S_{k-1}+S_{k-1}^2}{(k-1)\,|X_k|-|S_{k-1}|+k(k-1)\,\varepsilon}\,I_{B_{n,k}\cap\bar{D}_k\cap E_k}$$

$$\leqslant \frac{3}{k} \mathrm{E}\, |X_k|\, I_{B_{n,k}\cap \bar{D}_k \cap E_k} + \frac{\alpha^2}{\varepsilon k^2} \mathrm{E}\, I_{B_{n,k}\cap \bar{D}_k \cap E_k} + \frac{4\alpha^2}{\varepsilon k^2} \mathrm{E}\, I_{B_{n,k}\cap \bar{D}_k \cap \bar{E}_k}$$

$$\leqslant 12\alpha\, \mathrm{E} \frac{|X_k|}{|S_{k-1}|+k\alpha} \frac{|X_k|}{|X_k|+k\varepsilon} I_{B_{n,k}\cap \bar{D}_k \cap E_k} +$$

$$+ \frac{\alpha^2}{\varepsilon k^2} \mathrm{E}\, I_{B_{n,k}\cap \bar{D}_k \cap E_k} + \frac{4\alpha^2}{\varepsilon k^2} \mathrm{E}\, I_{B_{n,k}\cap \bar{D}_k \cap \bar{E}_k}$$

$$\leqslant 12\alpha\, \mathrm{E} \frac{X_k^2}{|X_k|\,|S_{k-1}|+k^2\varepsilon\alpha} I_{B_{n,k}\cap \bar{D}_k \cap E_k} +$$

$$+ \frac{\alpha^2}{\varepsilon k^2} \mathrm{E}\, I_{B_{n,k}\cap \bar{D}_k \cap E_k} + \frac{4\alpha^2}{\varepsilon k^2} \mathrm{E}\, I_{B_{n,k}\cap \bar{D}_k \cap \bar{E}_k}.$$

Choosing now k such that $\alpha/k < \varepsilon$ and using (6)–(8) we obtain (5).

By the Lemma and the considerations given above, we have

THEOREM 2. *Let $\{X_k\}$ be a sequence of integrable random variables. Then, for every $\alpha > 0$ and any given $\varepsilon > 0$,*

$$\left|\alpha P(B(\alpha)) - \sum_{k=n+1}^{\infty} \mathrm{E} \frac{|S_{k-1}|}{k-1} I_{B_{n,k}}\right|$$

$$\leqslant 6\varepsilon + 6\varepsilon \sum_{k=n}^{\infty} \mathrm{E} \frac{X_k}{k^2\varepsilon^2+X_k^2} I_{B_{n,k}} + 12\alpha \sum_{k=n}^{\infty} \mathrm{E} \frac{X_k^2}{k^2\alpha\varepsilon+|X_k|\,|S_{k-1}|} I_{B_{n,k}} + \\ + \frac{2n\alpha}{n-1} \mathrm{E} \frac{|S_{n-1}|}{(n-1)\alpha+|S_{n-1}|} + \mathrm{E} \frac{|S_{n-1}|}{n-1} I_{[|X_n| \geqslant n\alpha/2]}.$$

Simpler (but only slightly more restrictive) estimates are contained in the following

THEOREM 3. *Let $\{X_k; k \geqslant 1\}$ be a sequence of random variables on a probability space $(\Omega, \mathscr{A}, P)$. Then, for every $\alpha > 0$,*

$$\text{(9)} \quad \overline{\lim_{n\to\infty}} \left|\alpha P(B(\alpha)) - \sum_{k=n+1}^{\infty} (k-1)^{-1} \mathrm{E}\,|S_{k-1}|\, I_{B_{n,k}}\right|$$

$$\leqslant 2 \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k} \mathrm{E} \frac{|S_k|^r}{|S_k|^r + k^r\varepsilon^r} |X_k|\, I_{B_{n,k}} + \\ + 2\alpha \overline{\lim_{n\to\infty}}\, \mathrm{E} \frac{|S_{n-1}|}{(n-1)\alpha+|S_{n-1}|} + \overline{\lim_{n\to\infty}}\, \mathrm{E} \left\{\frac{|S_{n-1}|}{n-1} I_{[|X_n| > n\alpha/2]}\right\}$$

for all $r \geqslant 0$, while under the assumption that $\mathrm{E}\,|X_k|^{r+1} < \infty$ for $k \geqslant 1$ and for some $r \geqslant 0$ we have, for any given $\varepsilon > 0$ and every $\alpha > 0$,

$$\text{(10)} \quad \overline{\lim_{n\to\infty}} \left|\alpha P(B(\alpha)) - \sum_{k=n+1}^{\infty} (k-1)^{-1} \mathrm{E}\,|S_{k-1}|\, I_{B_{n,k}}\right|$$

$$\leqslant \varepsilon + 2^{r+1} \varepsilon^{-r} \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} k^{-(r+1)} \mathrm{E} |X_k|^{r+1} I_{B_{n,k}} +$$

$$+ 2\alpha \overline{\lim_{n\to\infty}} \mathrm{E} \frac{|S_{n-1}|}{(n-1)\alpha + |S_{n-1}|} + \overline{\lim_{n\to\infty}} \mathrm{E} \frac{|S_{n-1}|}{n-1} I_{[|X_n| \geqslant n\alpha/2]}.$$

Proof. First we prove (9). Note that

$$\overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k(k-1)} \mathrm{E} \left(\frac{|S_k|^r}{|S_k|^r + k^r \alpha^r} |(k-1) X_k - S_{k-1}| I_{B_{n,k}} \right)$$

$$\leqslant \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k} \mathrm{E} \frac{|S_k|^r}{|S_k|^r + k^r \alpha^r} |X_k| I_{B_{n,k}} +$$

$$+ \alpha \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k} \mathrm{E} \frac{|S_k|^r}{|S_k|^r + k^r \alpha^r} I_{B_{n,k}}$$

$$= \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k} \mathrm{E} \frac{|S_k|^r}{|S_k|^r + k^r \alpha^r} |X_k| I_{B_{n,k}},$$

which by (2′) and Theorem 2 gives (9).

To prove (10) it is enough to see that

$$\overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k(k-1)} \mathrm{E} \frac{|(k-1) X_k - S_{k-1}|^{r+1}}{|(k-1) X_k - S_{k-1}|^r + k^r (k-1)^r \alpha^r} I_{B_{n,k}}$$

$$\leqslant 2^{r+1} \alpha^{-r} \left(\overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{\mathrm{E} |X_k|^{r+1}}{k^{r+1}} I_{B_{n,k}} + \alpha \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k^{r+1}} \mathrm{E} I_{B_{n,k}} \right)$$

$$= 2^{r+1} \alpha^{-r} \overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{\mathrm{E} |X_k|^{r+1}}{k^{r+1}} I_{B_{n,k}}.$$

Hence, by (3′) and the observation before Theorem 2, we get (10).

Remark 2. It is not difficult to see that (9) and (10) do not require the convergence in mean to prove the strong law of large numbers. Moreover, we see that if $\{X_k;\ k \geqslant 1\}$ is a sequence of independent random variables, then

$$\overline{\lim_{n\to\infty}} \mathrm{E} \frac{|S_{n-1}|}{n-1} I_{[|X_n| \geqslant n\alpha/2]} = \overline{\lim_{n\to\infty}} P \left[|X_n| \geqslant \frac{n\alpha}{2} \right] \mathrm{E} \frac{|S_{n-1}|}{n-1},$$

which together with (9) or (10) leads us to conditions for the strong law of large numbers expressed in terms of moments conditions and convergence in probability.

As a conclusion of (9) we get

COROLLARY 2. *If* $\{X_k;\ k \geqslant 1\}$ *is a sequence of independent identically distributed random variables with* $\mathrm{E}X_1 = 0$, *then*

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} (k-1)^{-1} \mathrm{E}|S_{k-1}| I_{B_{n,k}} = 0.$$

Proof. It is enough to see that under the assumption of Corollary 2 we have $P(B(\alpha)) = 0$, $\alpha > 0$ (by Kolmogorov's strong law of large numbers), $\lim_{n\to\infty} n^{-1} \mathrm{E}|S_n| = 0$, and

$$\overline{\lim_{n\to\infty}} \sum_{k=n}^{\infty} \frac{1}{k} \mathrm{E} \frac{|S_k|^r}{|S_k|^r + k^r \varepsilon^r} |X_k| I_{B_{n,k}} \leqslant \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=n}^{\infty} \mathrm{E}|X| I_{B_{n,k}} = 0.$$

Remark 3. The same conclusion is true for $\{X_k;\ k \geqslant 1\}$ being ergodic.

By (10) we have

THEOREM 4. *Let* $\{X_k;\ k \geqslant 1\}$ *be a sequence of random variables* (*not necessarily independent*) *with* $\mathrm{E}X_k = 0$, $k \geqslant 1$, *and such that* (1′) *holds and*

$$\sum_{k=1}^{\infty} \mathrm{E}|X_k|^{r+1}/k^{r+1} < \infty \quad \textit{for some } r \geqslant 0.$$

Then $\{X_k;\ k \geqslant 1\}$ *satisfies the strong law of large numbers if and only if*

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} (k-1)^{-1} \mathrm{E}|S_{k-1}| I_{B_{n,k}} = 0. \tag{11}$$

COROLLARY 3. *If* $\{X_k;\ k \geqslant 1\}$ *is a sequence of independent random variables with* $\mathrm{E}X_k = 0$, $k \geqslant 1$, *and*

$$\sum_{k=1}^{\infty} \mathrm{E}|X_k|^{r+1}/k^{r+1} < \infty \quad \textit{for some } r \ (0 \leqslant r \leqslant 1),$$

then (11) *holds.*

Let $\{f_k\}$ be now a sequence of integrable functions on Ω such that $\int |f_k| \to 0$ and $|f_k - f_{k-1}| \to 0$ a.e. as mentioned previously. If $|f_k - f_{k-1}| \leqslant g \in L^1(\mu)$, then Theorem 1 gives a necessary and sufficient condition for a.e. convergence. This last condition is too strong, as it is not satisfied for many important sequences in a.e. convergence, e.g. the sequences in Birkhoff's ergodic theorem, i.e.

$$f_k = T_k f = \frac{1}{k} \sum_{j=0}^{k-1} T_\varphi^j f,$$

where $f \in L^1(\mu)$ while $T_\varphi f = f(\varphi)$, and φ is measure preserving. We now state a theorem which handles the above case and can be applied quite generally to positive operators. In the following, we put

$$B_{n,k}(\alpha) = [T_n f \leqslant \alpha,\ T_{n+1} f \leqslant \alpha, \ldots, T_k f > \alpha],$$

i.e. without using the absolute values.

THEOREM 5. *Let* $\{T_k\}$ *be a sequence of bounded operators on* $L^1(\mu)$. *Suppose*

$$T_k f \to 0 \ \textit{in} \ L^1(\mu) \quad \textit{and} \quad |T_{k+1} f - T_k f| \to 0 \ \textit{a.e.}$$

(a) *If there exists an integrable function g such that $T_k f \leqslant g$ for each k, then*

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} \int_{B_{n,k}} T_k f\, d\mu = \alpha\mu(B(\alpha)).$$

(b) *If there exists an integrable function g such that $T_k f \geqslant g$, then*

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} \int_{B_{n,k}} T_{k-1} f\, d\mu = \alpha\mu(B(\alpha)).$$

Proof. We prove only part (b). Let

$$h_n = \sum_{k=n+1}^{\infty} I_{B_{n,k}(\alpha)} T_{k-1} f + \alpha I_{B_{n,n}(\alpha)} \leqslant \alpha I_{B_n(\alpha)},$$

as $T_{k-1} f \leqslant \alpha$ on $B_{n,k}(\alpha)$ for $k \geqslant n+1$. Hence

$$\overline{\lim_{n\to\infty}}\, h_n \leqslant \alpha I_{B(\alpha)} \qquad (B_n(\alpha) \downarrow B(\alpha)).$$

But also

$$h_n = \sum_{k=n+1}^{\infty} I_{B_{n,k}(\alpha)} (T_{k-1} f - T_k f) + \sum_{k=n+1}^{\infty} I_{B_{n,k}(\alpha)} T_k f + \alpha I_{B_{n,k}(\alpha)}.$$

Since $|T_{k-1} f - T_k f| \to 0$ a.e. and $T_k f > \alpha$ on $B_{n,k}(\alpha)$, we have

$$\underline{\lim_{n\to\infty}}\, h_n \geqslant \alpha I_{B(\alpha)}$$

and, consequently,

$$\lim_{n\to\infty} h_n = \alpha I_{B(\alpha)} \text{ a.e.}$$

But for every n we obtain $h_n \leqslant \alpha I_{B_n(\alpha)} \leqslant \alpha$ and $h_n \geqslant g$. Thus

$$\begin{aligned}
\lim_{n\to\infty} \int h_n\, d\mu &= \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \int (I_{B_{n,k}(\alpha)} T_{k-1} f + \alpha I_{B_{n,n}(\alpha)})\, d\mu \\
&= \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \int_{B_{n,k}(\alpha)} T_{k-1} f\, d\mu + \alpha \int_{B_{n,n}(\alpha)} d\mu \\
&= \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \int_{B_{n,k}(\alpha)} T_{k-1} f\, d\mu = \lim_{n\to\infty} \sum_{k=n}^{\infty} \int_{B_{n,k}(\alpha)} T_{k-1} f\, d\mu.
\end{aligned}$$

As an example, let $T_k f = S_k f - Pf$, where S_k is a positive operator and $f \geqslant 0$. Assume $\int T_k f\, d\mu \to 0$ so that Pf is integrable. Then $T_k f \geqslant -Pg \in L^1(\mu)$, whence

$$\mu(\overline{\lim_{k\to\infty}}\, S_k f - Pf > 0) = 0 \quad \text{iff} \quad \lim_{n\to\infty} \sum_{k=n}^{\infty} \int_{B_{n,k}(\alpha)} T_{k-1} f\, d\mu = 0$$

for each $\alpha > 0$. Similarly, replacing f by $-f$ and changing $B_{k,n}$ suitably, we obtain $T_k(-f) \leqslant Pg$. Therefore

$$\mu\left(\underline{\lim}_{k\to\infty} T_k f < 0\right) = \mu\left(\overline{\lim}_{k\to\infty} T_k(-f) > 0\right) = 0$$

iff

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} \int_{B_{n,k}(\alpha)} T_k f \, d\mu = 0 \quad \text{for each } \alpha > 0.$$

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