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ON QUASI-CONTINUOUS ITERATION SEMIGROUPS AND GROUPS OF REAL FUNCTIONS

BY

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In this note we give a characterization of iteration semigroups and groups (semiflows and flows) defined in an interval $I \subset \boldsymbol{R}$ and continuous with respect to iterative parameter. First let us introduce the following

DEFINITIONS (cf. [5], [3], [2]). A family of functions $\{f^t, t \geqslant 0\}$ $[\{f^t, t \in \boldsymbol{R}\}]$ mapping I into itself is said to be an *iteration semigroup* [*group*] if

$$f^t \circ f^s = f^{t+s} \quad \text{for } t, s \geqslant 0 \ [t, s \in \boldsymbol{R}].$$

If for every $x \in I$ the mapping $t \to f^t(x)$ is continuous, then the iteration semigroup [group] is said to be *quasi-continuous* (abbreviated to q.i.s. [q.i.g.]). If, moreover, all functions f^t are continuous, then the quasi-continuous iteration semigroup [group] will be called *continuous* (in abbreviation c.i.s. [c.i.g.]).

1. In [5], [6] and [1] a characterization and the general construction of c.i.s.'s and c.i.g.'s are given. On the base of these results we give a characterization of q.i.s.'s and q.i.g.'s. We begin with the following lemmas:

LEMMA 1 (Sklar [4]). *Let $\{f^t, t \geqslant 0\}$ be a q.i.s. defined in an interval I and let x_0 be in I. If there exists an $s > 0$ such that $f^s(x_0) = x_0$, then $f^t(x_0) = x_0$ for all $t \geqslant 0$.*

LEMMA 2 (Sklar [4]). *Let $\{f^t, t \geqslant 0\}$ be a q.i.s. defined in I. Then, for any $x \in I$, either the function $t \to f^t(x)$ is strictly monotonic on the whole interval $\langle 0, \infty)$ or there exists an $s \geqslant 0$ such that $t \to f^t(x)$ is strictly monotonic on $\langle 0, s \rangle$, but $f^t(x) = f^s(x)$ for all $t \geqslant s$.*

LEMMA 3. *If $\{f^t, t \geqslant 0\}$ is an iteration semigroup defined in I, then*

$$f^t[I] = f^t[f^0[I]] \quad \textit{and} \quad f^t[I] \subset f^0[I] \quad \textit{for } t \geqslant 0.$$

The proof is trivial.

Now we prove the following

THEOREM 1. *An iteration semigroup $\{f^t, t \geqslant 0\}$ is quasi-continuous iff there exists a family of disjoint intervals $\{I_n, n \in M\}$, where $0 \leqslant \operatorname{card} M \leqslant \aleph_0$, such that*

$$(1)\qquad f^0[I] = \bigcup_{n\in M} I_n \cup \{x\in I\colon \bigwedge_{t\geq 0} f^t(x) = x\}\ ([1])$$

and, for every $n\in M$, $\{f^t|I_n,\ t\geq 0\}$ is a c.i.s. such that all $f^t|I_n$ are increasing but can be constant only in neighbourhoods of its fixed points.

Proof. Let $\{f^t,\ t\geq 0\}$ be a q.i.s. Put

$$A := \{x\in I\colon f^1(x) = x\},$$

$$h_x(t) := f^t(x), \qquad x\in I,\ t\geq 0,$$

$$J_x := h_x[\langle 0, \infty)], \qquad x\in I.$$

It is clear that $A\subset f^1[I]$, so $A\subset f^0[I]$ by Lemma 3. In view of Lemma 1 it follows that

$$A = \{x\in I\colon \bigwedge_{t\geq 0} f^t(x) = x\}.$$

Assume that $\bar{x}\in f^0[I]\setminus A$. The continuity and monotonicity of $h_{\bar{x}}$ imply that $J_{\bar{x}}$ is a proper interval with one end equal to $\bar{x}$, since $h_{\bar{x}}(0) = \bar{x}$. Note that by Lemma 3 it follows that $J_{\bar{x}}\subset f^0[I]$. First we shall show that, for every $t\geq 0$, $f^t|J_{\bar{x}}$ is continuous and increasing. Indeed, let $y\in J_{\bar{x}}$. Then there exists an $s\geq 0$ such that $y = f^s(\bar{x})$, so

$$f^t(y) = f^t(f^s(\bar{x})) = f^{t+s}(\bar{x}) = h_{\bar{x}}(t+s).$$

By Lemma 2 the following two cases may occur:

(i) $h_{\bar{x}}$ is strictly monotonic in $\langle 0, \infty)$. Then $s = h_{\bar{x}}^{-1}(y)$ and

$$(2)\qquad f^t(y) = h_{\bar{x}}(t+h_{\bar{x}}^{-1}(y)) \quad \text{for } t\geq 0 \text{ and } y\in J_{\bar{x}}.$$

(ii) h_x is strictly monotonic in $\langle 0, \gamma\rangle$ for a $\gamma > 0$ and $h_{\bar{x}}(t) = h_{\bar{x}}(\gamma)$ for $t\geq\gamma$. We may assume that $s\in\langle 0, \gamma\rangle$. Then

$$s = (h_{\bar{x}}|\langle 0, \gamma\rangle)^{-1}(y)$$

and

$$(3)\qquad f^t(y) = h_{\bar{x}}(t+(h_{\bar{x}}|\langle 0, \gamma\rangle)^{-1}(y)) \quad \text{for } t\geq 0 \text{ and } y\in J_{\bar{x}}.$$

Hence in both cases $f^t|J_{\bar{x}}$ is continuous, increasing and

$$f^t[J_{\bar{x}}]\subset J_{\bar{x}} \quad \text{for } t\geq 0.$$

Put

$$L_x := \{y\in f^0[I]\colon J_x\subset J_y\}$$

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[1] If $M = \emptyset$, then $\bigcup_{n\in M} I_n := \emptyset$.

and

$$(4) \qquad I^x := \bigcup_{y \in L_x} J_y \quad \text{for } x \in f^0[I] \setminus A.$$

Let us note that, for every $x \in f^0[I]\setminus A$, I^x is an interval, $I^x \subset f^0[I]$ and $\{f^t|I^x,\ t \geqslant 0\}$ is a c.i.s. such that all $f^t|I^x$ are increasing.

The intervals I^x have the following property: for any $x, y \in f^0[I]\setminus A$ either $I^x = I^y$ or $I^x \cap I^y = \emptyset$ or else $I^x \cap I^y$ is a singleton. In fact, the interval J_x has one of the following forms: $(p_x, x\rangle$, $\langle p_x, x\rangle$, $\langle x, p_x)$ or $\langle x, p_x\rangle$, where

$$p_x := \lim_{t\to\infty} h_x(t).$$

From (2) and (3) we infer that $f^t(p_x+) = p_x$ or $f^t(p_x-) = p_x$ and $f^t(p_x) = p_x$ if $p_x \in J_x$ and, moreover, the difference $f^t(y) - y \neq 0$ has constant sign for $y \in J_x\setminus\{p_x\}$, $t > 0$. Hence by the continuity of $f^t|J_x$ it follows that, for any $x, y \in f^0[I]\setminus A$, $p_x \notin \operatorname{Int} J_y$. Therefore the following three cases may occur:

(i) $J_x \subset J_y$ or $J_y \subset J_x$; then $p_y = p_x$ and $I^y = I^x$;

(ii) $J_x \cap J_y = \emptyset$; then $I^x \cap I^y = \emptyset$;

(iii) $J_x \cap J_y = \{p\}$; then $p_x = p_y = p$ and $I^x \cap I^y = \{p\}$.

Let us note that $p_x \in J_x$ iff h_x is constant in a neighbourhood of ∞. Hence (2) and (3) imply that $p_x \in J_x$ iff every $f^t|J_x$ is constant in a neighbourhood of p_x. In view of (4) it follows that also the interval I^x has the above property and, in consequence, if $I^x \cap I^y = \{p\}$, then $I^x \cup I^y$ is an interval, $p = p_x = p_y \in \operatorname{Int}(I^x \cup I^y)$ and every function $f^t|I^x \cup I^y$ is continuous. Moreover, every function f^t is constant in a neighbourhood of p and

$$f^t[I^x \cup I^y] \subset I^x \cup I^y \quad \text{for } t \geqslant 0.$$

Let $\{I_n,\ n \in M\}$ be a set of intervals such that $I_n \neq I_m$ for $n \neq m$ and each I_n is either equal to I^x with the property that, for every $y \in f^0[I]\setminus A$, $I_x \cap I_y = \emptyset$ or $I_x = I_y$, or is a union of two intervals I^x and I^y such that

$$\operatorname{card}(I^x \cap I^y) = 1 \quad \text{and} \quad \bigcup_{n \in M} I_n = \bigcup_{x \in f^0[I]\setminus A} I^x.$$

If $A = f^0[I]$, we take $M = \emptyset$. Obviously, the intervals I_n, $n \in M$, are pairwise disjoint and $\operatorname{card} M \leqslant \aleph_0$.

For every $n \in M$, $I_n \subset f^0[I]$, since all $J_x \subset f^0[I]$, so

$$\bigcup_{n \in M} I_n \cup A \subset f^0[I].$$

On the other hand, if $x \in f^0[I]\setminus A$, then there exists an $n \in M$ such that $J_x \subset I_n$. Hence $x \in I_n$ and, consequently, relation (1) holds.

By the definition of I_n and the properties of the functions f^t proved above it follows that $\{f^t|I_n,\ t \geqslant 0\}$ is a c.i.s. and $f^t|I_n$ are increasing but can be constant only in neighbourhoods of their fixed points.

The sufficient condition is an immediate consequence of the property that, for every $x \in I$, $f^t(x) = f^t(f^0(x))$ and either $f^0(x) \in I_n$ for an $n \in M$ or $f^t(f^0(x)) = f^0(x)$ for $t \geqslant 0$.

THEOREM 2. *An iteration group* $\{f^t,\ t \in \mathbf{R}\}$ *is quasi-continuous iff there exists a family of disjoint intervals* $\{I_n,\ n \in M\}$ *such that relation* (1) *holds and, for every* $n \in M$, $\{f^t|I_n,\ t \in \mathbf{R}\}$ *is a c.i.g. such that all* $f^t|I_n$ *are strictly increasing functions. Moreover, then* $f^t[I_n] = I_n$ *for* $t \in \mathbf{R}$.

Proof. Let $\{f^t,\ t \in \mathbf{R}\}$ be a quasi-continuous iteration group. Consider the restriction of the group $\{f^t,\ t \in \mathbf{R}\}$ to the semigroup $\{f^t,\ t \geqslant 0\}$. Let $\{I_n,\ n \in M\}$ be a family of intervals determined in the previous theorem for the semigroup $\{f^t,\ t \geqslant 0\}$. For every $t \in \mathbf{R}$,

$$f^t[f^0[I]] = f^0[f^t[I]] \subset f^0[I],$$
$$f^0(x) = x \quad \text{for } x \in f^0[I]$$

and

$$f^t \circ f^{-t} = f^{-t} \circ f^t = f^0.$$

Hence every function $f^t|f^0[I]$ is invertible and

$$f^{-t}|f^0[I] = f^t|f^0[I]^{-1}.$$

Hence in view of Theorem 1 and the construction of the intervals I_n it follows that $f^t|I_n$ for $t \geqslant 0$ are strictly increasing and $I_n \cap A = \emptyset$ for every $n \in M$. Let $x \in I_n$ and $t > 0$; then $f^{-t}(x) \in I_n$. In fact,

$$f^{-t}(x) \in f^0[I] \quad \text{and} \quad f^{-t}(x) \notin A,$$

since $x \notin A$. By (1), $f^{-t}(x) \in I_m$ for an $m \in M$. Note that $m = n$, since $I_n \cap I_k = \emptyset$ for $k \neq n$ and

$$x = f^t(f^{-t}(x)) \in f^t[I_m] \subset I_m.$$

Thus $f^{-t}[I_n] \subset I_n$. Hence in view of Theorem 1 we get the necessary condition. We have proved that, for every $t \in \mathbf{R}$ and $n \in M$, $f^t[I_n] \subset I_n$. Hence $I_n \subset f^{-t}[I_n]$ for $t \in \mathbf{R}$ and $n \in M$, since $f^0(x) = x$ for $x \in I_n$ and, consequently, $f^t[I_n] = I_n$ for $t \in \mathbf{R}$ and $n \in M$.

The sufficient condition is obvious.

Remark 1. If $\{f^t,\ t \geqslant 0\}$ $[\{f^t,\ t \in \mathbf{R}\}]$ is a q.i.s. [q.i.g.], then formula (1) in the necessary condition of Theorem 1 [Theorem 2] may be replaced by

$$f^0[I] = \bigcup_{n \in M} I_n \cup \{x \in I\colon f^1(x) = x\}. \tag{5}$$

However, the converse is not true. Consider the following example:

Let $I = \langle 0, 1\rangle$ and $f^t(x) := x+t \pmod{1}$, $t \in \boldsymbol{R}$. The family $\{f^t,\ t\in \boldsymbol{R}\}$ is an iteration group which is not quasi-continuous and satisfies relation (5) with $M = \emptyset$.

Now on the base of Theorems 1 and 2 we give the general construction of quasi-continuous iteration semigroups and groups. The general form of continuous iteration semigroups and groups is well known (see [5], [6], [1]). We use this fact in the following

Construction of quasi-continuous iteration semigroups and groups.

1° Let $\{I_n,\ n\in M\}$ be a sequence of pairwise disjoint intervals contained in I (we admit $M = \emptyset$).

2° Let $\{g_n^t,\ t \geqslant 0\}$ $[\{g_n^t,\ t\in \boldsymbol{R}\}]$ be a c.i.s. [c.i.g.] in I_n such that all functions g_n^t are increasing and $g_n^0(x) = x$ for $x \in I_n$.

3° Let A be an arbitrary subset of $I \setminus \bigcup_{n\in M} I_n$.

4° Put

$$J := \bigcup_{n\in M} I_n \cup A$$

and let a be an arbitrary function defined in I such that $a[I] = J$ and $a(x) = x$ for $x \in J$.

5° Put

$$f^t(x) = \begin{cases} a(x), & x\in a^{-1}[A],\ t\geqslant 0\ [t\in \boldsymbol{R}], \\ g_n^t(a(x)), & x\in a^{-1}[I_n],\ t\geqslant 0\ [t\in \boldsymbol{R}]. \end{cases} \tag{6}$$

Formula (6) gives the general form of quasi-continuous iteration semigroups and groups.

Proof. It is straightforward to verify that the family of functions f^t defined above is a quasi-continuous iteration semigroup [group].

Conversely, let $\{f^t,\ t\geqslant 0\}$ $[\{f^t,\ t\in \boldsymbol{R}\}]$ be a q.i.s. [q.i.g.]. Put

$$J = f^0[I] \quad \text{and} \quad A = \{x\in I\colon \bigwedge_{t\geqslant 0} f^t(x) = x\}.$$

Let $\{I_n,\ n\in M\}$ be a sequence of disjoint intervals determined in Theorem 1 [Theorem 2], $g_n^t := f^t|I_n$ for $t\geqslant 0$ $[t\in \boldsymbol{R}]$ and $a = f^0$. We have

$$J = \bigcup_{n\in M} I_n \cup A.$$

Hence Theorem 1 [Theorem 2] implies that f^t is of the form (6).

Remark 2. The general form of a c.i.g. in an interval I is the following: Let $K \subset I$ be an interval relatively closed in I and $a\colon I \to K$ be a continuous function such that $a(x) = x$ for $x\in K$. Let $\{g^t,\ t\in \boldsymbol{R}\}$ be a c.i.g. in K such that all

g^t are strictly increasing in K (then $g^0(x) = x$ in K and $g^t[K] = K$ for $t \in \mathbf{R}$) — the general construction of such groups $\{g^t,\ t \in \mathbf{R}\}$ is well known (see [6] and [1]). Then $f^t(x) = g^t(a(x))$, $t \in \mathbf{R}$, $x \in I$, is a c.i.g. and every c.i.g. is of this form.

2. Now we shall consider the problem of the embeddability of a given function into quasi-continuous iteration semigroups and groups. That is, we shall characterize functions f for which there exists a q.i.s. $\{f^t,\ t \geqslant 0\}$ or a q.i.g. $\{f^t,\ t \in \mathbf{R}\}$ such that $f^1 = f$.

THEOREM 3. *The function f: $I \to I$ is embeddable into a q.i.s. iff there exists a family of disjoint intervals $\{I_n \subset I,\ n \in M\}$ such that*

$$f[I] = \bigcup_{n \in M} f[I_n] \cup \{x \in I: f(x) = x\} \tag{7}$$

and, for every $n \in M$, $f[I_n] \subset I_n$, and $f|I_n$ is continuous and increasing but can be constant only in neighbourhoods of its fixed points.

Proof. Let $\{f^t,\ t \geqslant 0\}$ be a q.i.s. such that $f^1 = f$ and let I_n, $n \in M$, be the intervals determined in Theorem 1. From (1) and Lemma 3 we get immediately formula (7) since, by Lemma 1,

$$f[\{x \in I: \bigwedge_{t \geqslant 0} f^t(x) = x\}] = \{x \in I: f(x) = x\}.$$

Moreover, from Theorem 1 it follows that $f[I_n] \subset I_n$, and $f|I_n$ is continuous and increasing. By the construction of the intervals I_n, $n \in M$, in the proof of Theorem 1 it follows that $f|I_n$ can be constant only in neighbourhoods of its fixed points.

Conversely, let f satisfy the assumptions given in the theorem. Put

$$A := \{x \in I: f(x) = x\} \quad \text{and} \quad J := \bigcup_{n \in M} I_n \cup A.$$

From Theorem 2 in [5] (or Theorem 9.1 in [6]) it follows that $f|I_n$ is embeddable into a c.i.s. $\{g_n^t,\ t \geqslant 0\}$ in I_n. Let a be an arbitrary function defined in I such that

$$a(x) = x \quad \text{for } x \in J$$

and

$$a(x) \in f^{-1}[\{f(x)\}] \cap J \quad \text{for } x \in J.$$

Then $f(a(x)) = f(x)$, $a[I] = J$. Let $\{f^t,\ t \geqslant 0\}$ be the iteration semigroup defined by formula (6). This semigroup is quasi-continuous and $f^1 = f$.

THEOREM 4. *The function f: $I \to I$ is embeddable into a q.i.g. iff there exists a family of disjoint intervals $\{I_n \subset I,\ n \in M\}$ such that*

(8) $$f[I] = \bigcup_{n\in M} I_n \cup \{x\in I\colon f(x) = x\}$$

and, for every $n\in M$, $f[I_n] = I_n$, *and* $f|I_n$ *is continuous and strictly increasing.*

Proof. Let f be embeddable into a q.i.g. $\{f^t,\ t\in \mathbf{R}\}$ and let $\{I_n,\ n\in M\}$ be the family of intervals determined in Theorem 2. By formula (1), Lemmas 1 and 3 and Theorem 3 we have

$$f[I] = f[f^0[I]] = \bigcup_{n\in M} f[I_n] \cup f[\{x\in I\colon \bigwedge_{t\geqslant 0} f^t(x) = x\}]$$

$$= \bigcup_{n\in M} I_n \cup \{x\in I\colon f(x) = x\}.$$

Moreover, from Theorem 2 it follows that $f[I_n] = I_n$ and $f|I_n$ are strictly increasing for $n\in M$.

Conversely, for every $n\in M$, $f|I_n$ is embeddable into a c.i.g. $\{g_n^t,\ t\in \mathbf{R}\}$ such that all g_n^t are strictly increasing (see [5]). Put $J := f[I]$. Obviously, f is invertible in J. The family of functions $\{f^t,\ t\in \mathbf{R}\}$ defined by (6) where $a(x) := f|J^{-1}(f(x))$ is a q.i.g. such that $f^1 = f$.

THEOREM 5. *If* $\{f^t,\ t\geqslant 0\}$ *is a q.i.s. defined in an interval* I *and* f^1 *is continuous in* I *and non-constant in neighbourhoods of its fixed points, then all* f^t *are continuous in* I.

Proof. Let $\{I_n,\ n\in M\}$ be the intervals determined in Theorem 1. Put

$$J := f^0[I], \quad f := f^1 \quad \text{and} \quad A := \{x\in I\colon f(x) = x\}.$$

First we shall show that J is an interval. We can write formula (1) in the following form:

(9) $$J = \bigcup_{n\in M} I_n \cup A.$$

Hence, by Lemma 3,

(10) $$f[I] = f[J] \quad \text{and} \quad f[J] \subset J,$$

since

(11) $$f[I_n] \subset I_n \quad \text{and} \quad f[A] = A.$$

By the continuity of f the set $K := f[I]$ is an interval. In view of (9)–(11) we have

$$K = \bigcup_{n\in M} f[I_n] \cup A \subset \bigcup_{n\in M} I_n \cup A,$$

whence

$$\bigcup_{n\in M} I_n \cup K \subset \bigcup_{n\in M} I_n \cup A.$$

On the other hand, $A \subset K$, so, by (9),

$$J = \bigcup_{n\in M} I_n \cup K. \tag{12}$$

For every $n \in M$, $K \cap I_n \neq \emptyset$, since $f[I_n] \subset f[I] \cap I_n$. Hence, by (12), J is an interval as a union of intervals which intersect the interval K. From the definition of the intervals I_n and formula (2) for $t = 1$ it follows that f is strictly increasing in I_n and, consequently, also in J. Thus

$$f^0(x) = f|J^{-1}(f(x)),$$

since $f(f^0(x)) = f(x)$ and $f^0(x) \in J$ for $x \in J$. Moreover, $f|J^{-1}$ is continuous since J is an interval and, consequently, f^0 is continuous.

Denote the ends of the interval J by a and b, and the ends of I_n by a_n and b_n. In view of the continuity of f it follows that either $a_n, b_n \in A$ or $a_n = a$ and $b_n \in A$ or else $a_n \in A$ and $b_n = b$. Hence Theorem 1, Lemma 2 and (9) imply the inclusion

$$f^t[\bar{I}_n \cap J] \subset \bar{I}_n \cap J \tag{13}$$

and the continuity of $f^t|I_n$ for $t \geqslant 0$. By Lemma 2 we have

$$|f^t(x) - x| \leqslant |f(x) - x| \quad \text{for } x \in I_n \text{ and } 0 < t < 1,$$

since $f^0(x) = x$ for $x \in I_n$. Hence the continuity of $f^t|I_n$ and Lemma 1 imply that $f^t|\bar{I}_n \cap J$ for $0 < t < 1$ are continuous. The ends of all intervals I_n except perhaps two of them belong to A. If the left (right) end a_n (b_n) of I_n does not belong to A, then $a_n = a$ ($b_n = b$). Hence by (9), (13), Lemma 1 and the continuity of $f^t|\bar{I}_n \cap J$ for $0 < t < 1$ it is easy to see that $f^t|J$ for $0 < t < 1$ are continuous. For every $t > 0$, $f^t = (f^{t/m})^m$, where $m = [t]+1$, hence all $f^t|J$ are continuous. Finally, the formula

$$f^t(x) = f^t|J(f^0(x)) \quad \text{for } x \in I,\ t \geqslant 0,$$

implies that all f^t are continuous in I.

From Theorems 4 and 5 we have the following

COROLLARY 1. *Let $f[I] =: K$. A continuous function f is embeddable into a q.i.g. iff $f[K] = K$ and $f|K$ is strictly increasing. Then every q.i.g. of f is continuous.*

In [6] the author has proved that in the definition of c.i.s.'s the continuity of the mappings $t \to f^t(x)$ for $x \in I$ can be replaced by the measurability of these mappings. The same is not true for quasi-continuous iteration semigroups. This is shown by the following

EXAMPLE. Let

$$f^t(x) = \begin{cases} 2^{-t}x, & t\in\langle 0, 1), \ x\in\langle 0, 2^{t-2}\rangle \text{ or } t\geqslant 1, \ x\in\langle 0, 2^{-1}\rangle, \\ 1-2^{-t}x, & t\in\langle 0, 1), \ x\in(2^{t-2}, 2^{-1}\rangle, \\ 1-2^{-t}(1-x), & t\in\langle 0, 1), \ x\in\langle 2^{-1}, 1-2^{t-2}), \\ 2^{-t}(1-x), & t\in\langle 0, 1), \ x\in\langle 1-2^{t-2}, 1) \text{ or } t\geqslant 1, \ x\in\langle 2^{-1}, 1\rangle. \end{cases}$$

It is easy to verify that $\{f^t, \ t\geqslant 0\}$ is an iteration semigroup with the following properties:

1° For $t\geqslant 1$, f^t are continuous.

2° For $0 < t < 1$, f^t have exactly two points of discontinuity.

3° For $x\in\langle 0, 1/4\rangle\cup\langle 3/4, 1\rangle$ the mappings $t\to f^t(x)$ are continuous.

4° For $x\in(1/4, 3/4)$ the mappings $t\to f^t(x)$ have exactly one point of discontinuity.

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