GENERALIZED FINITE PLANES

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This paper deals with systems equivalent to Szamkołowicz's finite regular planes (see [4]). With point and line being undefined terms and incident on (or merely "on") being an undefined relation, a collection of lines and points satisfying the following axioms (for given m and n) is called a *plane*:

1. If P and Q are distinct points, there is exactly one line λ such that P is on λ and Q is on λ (if a point is one a line, it will also be said that the line is on the point).

2. There exists at least one line; every line has exactly n points on it, $n \geqslant 2$, and if λ is any line, there is at least one point not on λ [1].

3. If P is any point not on a line λ, then there are exactly m lines on P, each of which is parallel to λ (i. e., each of which has no point on it which is also on λ).

The generalization, of course, is with this axiom 3. If $m = 0$, the axioms describe finite projective planes; if $m = 1$, they describe finite affine planes. For $m \geqslant 2$, they define a new class of Bolyaí-Lobatchevski planes, different from those of Graves [2] and Topel [5], but equivalent to those of Szamkołowicz [4]. Three theorems are proved with relatively little effort and are given here without complete proofs:

I. *Every point has exactly* $n+m$ *lines on it.*

(Axiom 2 implies that, given any point P, there is a line λ not on it. On P are n lines incident on the n points of λ and m lines parallel to λ; there are no other lines on P.)

II. *There are exactly* $t = (n+m)(n-1)+1$ *points in a plane.*

III. *If* q *is the number of lines in a plane, then* $qn = t(n+m)$.

A necessary condition for the existence of a plane is that n divides $m(m-1)$. To prove this, it is only necessary to consider condition III.

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[1] The author has shown [3] that this axiom can be weakened to read " There exists at least one line with exactly n points on it, $n \geqslant 2, \ldots$".

Dividing by n gives

$$q = \frac{(n+m)t}{n} = (n+m)^2 - n - 2m + 1 - \frac{m^2 - m}{n}.$$

Since q is an integer, $(m^2 - m)/n$ must be an integer, from which the result follows. An immediate consequence of this is that if $m = 2$, n can only be 2.

A sufficient condition for the existence of a plane is that $n = 2$, since any collection of t distinct objects, $t \geqslant 3$, satisfies the axioms of a plane when any pair of distinct objects is called a line and any object is called a point (a point is "on" a line if and only if it is one of the two objects making up that line). A model isomorphic to such a collection is a convex polygon with all of its diagonals (with the proper interpretation of points, lines, and "on"). Hence there exist planes for any m, if $n = 2$.

A sufficient condition is that n is a power of a prime, and that m is suitably chosen as follows: consider finite affine k-space with n points on a line. Let f be the number of lines on a point P of this space. Then if λ is a line not on P, there are n lines on P and on points of λ, so there are exactly $f - n$ lines parallel to the given line λ, on P, if the entire k-space is considered as a plane. For $m = f - n$, n the chosen power of a prime, this is a model of the axioms. An infinity of similar models can be constructed beginning with projective k-space in place of affine. When n is a prime, the models derived from the use of affine k-space may be represented in the following way:

Point: an ordered s-tuple of residue classes of integers modulo n.

Line: the range of a function of the type $L(S) = a + bS$, where a and b are "points", $b \neq (0, 0, \ldots, 0)$, S is an integer, and the operations are done modulo n.

A point is on a line: the point is an element of the line.

It is not a difficult exercise to show that the collection of all such points and lines is a plane in which

$$m = \frac{n^s - 1}{n - 1} - n.$$

So such planes exist. The question might be asked: for given m and n, are the axioms categorical?

Of course, for $m = 0$, or $m = 1$, the negative answer is shown by the existence of more than one projective plane and more than one affine plane (non-isomorphic) for certain values of n. And the answer is negative for $m > 2$ also; for example, in the case $n = 3$, $m = 3$, two non-iso-

morphic models exist. This has been demonstrated by E. Witt in an elegant discussion of Steiner systems ([6], p. 270).

The models and a seemingly new proof that they are non-isomorphic follow.

Consider points as letters and lines as triples of letters; then this collection of lines and points satisfies the axioms (a point is on a line, of course, if it is an element of the triple representing the line):

ABC	*BDE*	*CDM*	*DFH*	*EFL*	*FKJ*	*IKM*
AJD	*BGH*	*CEJ*	*DGK*	*EIG*	*GMJ*	
AHI	*BKL*	*CFI*	*DIL*	*EHM*	*HLJ*	
AEK	*BFM*	*CHK*				
AGF	*BJI*	*CGL*				
AML						

Model A

The following is also a model (points are integers, lines are triples of integers, and "on" means "is an element of"):

1	2	5	1	3	8
2	3	6	2	4	9
3	4	7	3	5	10
4	5	8	4	6	11
5	6	9	5	7	12
6	7	10	6	8	13
7	8	11	7	9	1
8	9	12	8	10	2
9	10	13	9	11	3
10	11	1	10	12	4
11	12	2	11	13	5
12	13	3	12	1	6
13	1	4	13	2	7

Model B

The reader should note that the order of elements in a triple is not critical in such models as these.

The non-isomorphism is demonstrated by the presence of a configuration of lines in *A* which cannot be duplicated by any renaming of points of *B*. The configuration is the set of lines and points *CBA*, *CGL*, *CJE*, *KMI*, *FHD*. This configuration can be represented schematically:
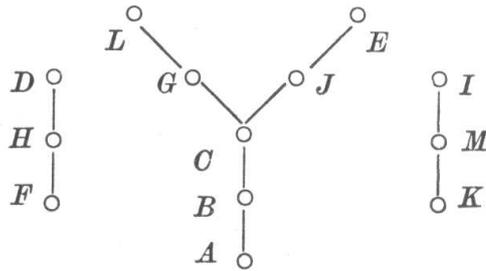
Fig. 1

The distinguishing feature of this configuration is that all 13 points of the plane are represented on three co-punctal lines and two parallel to each other and to each of the three copunctal lines. To show that a set of lines fulfilling these conditions does not exist in B, it is only necessary to look at lines parallel to one line, say 1 2 5, and to do the same for the line 1 3 8. To demonstrate the process, the lines on 3 which are parallel to 1 2 5 are considered. They are 3 4 7, 3 12 13, and 3 11 9. No line is parallel to them and to 1 2 5, so the configuration cannot be duplicated in this case. For the lines on 4 parallel to 1 2 5, the same situation occurs, as it does for 6, 7, ..., 13 (this is not obvious, but can be verified by listing the parallels in each case). The corresponding statements are true for any line in the 1 2 5 column. The same process, with 1 3 8 in place of 1 2 5, is carried out, with the same result. Hence, no set of lines with the incidences described occurs in B, and no renaming of points will effect a configuration as described, so B is not isomorphic to A.

Models can be constructed by various methods, if the models exist. They can, for instance, be constructed by a trial and error method, starting with a line and adding other points and lines until the complete plane is constructed (or all possibilities have been exhausted). A description of a less tedious method of construction will follow this lemma:

LEMMA. *Let π be a set of points and lines subject to the conditions:*

(1) there is at most one line on two distinct points, and every line has n points on it ($n \geqslant 2$),

(2) there exist exactly u points, $u = (n+k)(n-1)+1$ for some integer $k \geqslant 0$,

(3) there exist exactly $[(n+k)/n] \cdot u$ lines;
then π is a plane.

Proof. Let any ordered pair (P, w) (P a point, w a line, of π) be called an incidence. We say $(Q, r) = (S, v)$ if and only if $Q = S$ and $r = v$. Since there are $(n+k)u/n$ lines each with n points on them, there are at least $(n+k)u$ incidences. If the point P_i ($i = 1, 2, ..., u$) has a_i

lines on it, there are $a_i(n-1)$ points on those lines other than the point P_i. But there are only $u-1$ points other than P_i in all of π, so

$$a_i(n-1) \leqslant u-1 = (n+k)(n-1) \quad \text{or} \quad a_i \leqslant n+k.$$

Further, the total number of incidences in π is $\sum_{i=1}^{u} a_i$. If $a_f < n+k$ for some f $(1 \leqslant f \leqslant u)$, we have

$$\sum_{i=1}^{u} a_i < \sum_{i=1}^{u} (n+k) = u(n+k),$$

a contradiction, since there are at least $u(n+k)$ incidences. Hence there are $n+k$ incidences in which P_i occurs, for any i. In other words, there are $n+k$ lines on any point of π.

Let P and Q be a pair of distinct points of π for which there is no line λ such that P is on λ and Q is on λ. Such pairs will be called *unjoined*. Pairs of distinct points for which there is a line such that both points are on it will be called *joined*. Let x be the number of unjoined pairs in π. Then x is the difference of the total number of distinct pairs of points and the number of joined pairs, or

$$x = \binom{u}{2} - \binom{n}{2} \frac{n+k}{n} u = \frac{u(u-1)}{2} - \frac{un(n-1)(n+k)}{2n}$$

$$= \frac{u}{2}[(n+k)(n-1)-(n-1)(n+k)] = 0.$$

So if P and Q are distinct points, there is a line λ such that P is on λ and Q is on λ, and axiom 1 holds. Axiom 2 is easily shown to be satisfied. Then suppose P is any point not on a given line λ. There are $n+k$ lines on P, exactly n of which are on points of λ. Hence there are exactly k lines on P which are parallel to λ, and axiom 3 is satisfied. This shows that π is a plane. In fact, what has been said up to this point implies the equivalence of conditions 1, 2, and 3 with the conditions of the axioms.

A method of constructing planes for which $n = 3$ and $n+m = hn$ will now be described (for certain integers h).

Let a circle C of circumference $t = (n+m)(n-1)+1$ be divided into t equal arcs by t appropriately located points on the circle. Let these points be numbered in a clockwise manner from 1 to t. The distance between two of these points is then defined to be the length of the shortest arc of C whose end points are the given points. The distance between two radii of C is defined to be the distance between their points of intersection with C. Finally, a dial is defined as a set of 3 radii of C subject to

these conditions: no pair is separated by the same distance as any other pair, and every pair is separated by an integral distance $(1, 2, 3, \ldots)$. Two dials are called *disjoint* if no pair of radii of one are separated by a distance equal to that which separates a pair of the other.

With C fixed, let a dial be rotated (clockwise, say) with the distance between each two radii kept fixed. The dial successively "points out" triples of integers representing points of intersection of the dial with C. Starting with (a, b, c), triples $(a+1, b+1, c+1), \ldots, (a+t-1, b+t-1, c+t-1)$ (additions $\bmod t$), are described. Hence every dial describes t distinct triples. We will now prove the following

THEOREM. *If h disjoint (pairwise) dials can be found in the above described way, then the ht triples generated by them are a model for the axioms (points: $1, 2, \ldots, t$; lines: triples of points).*

Proof. We will show that the conditions of the lemma are satisfied. With $k = m$ and $n = 3$, condition 2 is satisfied. $(n+m)t/n = hnt/n = ht$ and there are ht lines, so condition 3 holds. Also, two points P, Q, on C are separated by a distance d, say. Any line on P and on Q is formed by a unique dial (F) which has two radii separated by distance d (F is unique, since dials are disjoint). The position of rotation of F is completely determined if one of the two radii specified is to be incident on P and the other on Q. The position of the third radius, and the point of C incident on it are then completely determined. The triple containing P and Q (if any has been formed) is thus completely determined. Hence there is at most one line on P and Q. There are three points on every line, so condition 1 is met. Therefore the set of triples described is a plane.

Planes such as these are determined by the set of k disjoint dials used in forming them; and each of these dials is associated with a triple of integers (s_1, s_2, s_3). This association can be made as follows: with a dial in a fixed position, the intersections of the three radii with C divide C into three arcs (a_1, a_2, a_3) in clockwise cyclic order. Let the actual lengths of these three arcs be s_1, s_2, s_3, respectively. Then (s_1, s_2, s_3) is a unique representation of the given dial; such representations will be referred to as *line cycles*.

The sum of the integers of a line cycle is always t. So the line cycle is, effectively, a partition of t into three unequal summands. These partitions also have the property (easily verified)

(x) if d is the sum of two summands of a partition and e is a summand or the sum of two other summands of the same partition, then $d \not\equiv e \pmod{t}$.

Partitions A and B associated with two disjoint dials have the additional property

(y) if d is a summand or the sum of two summands of A and e is the summand or the sum of two summands of B, then $d \not\equiv e \pmod{t}$.

Then if k partitions exist, each with property (x), and each pair of distinct partitions with property (y), the plane exists and can be generated by means of those partitions.

A necessary condition for the existence of planes for which $n = 3$ is that $m(m-1)/3$ is an integer. Of two consecutive integers, at most one is a multiple of 3, so 3 may divide m or $m-1$, but not both. Suppose 3 divides m. Then $m = 3k$ or $m+3 = 3(k+1)$, and the plane can be constructed by the process just discussed (if the appropriate partitions can be found). If 3 divides $m-1$, $m = 3k+1$. In this case, the number of lines to be constructed is

$$q = (n+m)t/n = (3+m)t/3 = [3(k+1)+1]t/3 = (k+1)t + t/3.$$

$k+1$ appropriate line cycles will construct $(k+1)t$ of these lines, and a pseudo-line cycle $Q = (t/3, t/3, t/3)$ will generate the other $t/3$. The requirement here on the $k+1$ line cycles is that they are pairwise disjoint and are associated with dials whose radii are never at a distance $t/3$ apart. Then it is easily verified that the line cycles and Q generate the required number of triples, and that they determine a plane (if these partitions exist) in the case 3 divides $m-1$.

Hence, in either case, a plane can be constructed if an appropriate set of partitions of t can be found (and $n = 3$). Table A gives a short list of examples calculated by hand.

TABLE A

m	t	q	Description by line cycles and pseudo-line cycles
0	7	7	(1 2 4)
3	13	26	(1 3 9) (2 5 6)
4	15	35	(1 3 11) (2 6 7) (5 5 5)
6	19	57	(1 3 15) (2 7 10) (5 6 8)
7	21	70	(1 2 18) (4 8 9) (5 6 10) (7 7 7)
9	25	100	(1 11 13) (2 6 17) (3 7 15) (4 5 16)
10	27	117	(1 2 24) (4 7 16) (5 10 12) (6 8 13) (9 9 9)

It should be mentioned that the set of triples obtained by taking $m = 4$ was reported by Barshop [1] as part of a proof of the independence of a set of axioms for a finite affine plane.

The affine plane ($m = 1$) admits of no representation by line cycles when $n = 3$. For $m = 2$, 5, and 8, the necessary condition is not met and no plane exists. For $m = 4$, the combinations of cycles shown are

the only ones possible. There are 3 for $m = 6$, 4 for $m = 7$, and 15 for $m = 9$. Whether these different possibilities always yield non-isomorphic planes or not has not yet been determined. For larger n, the order of summands chosen is critical and other slight modifications in the procedure are necessary. For $n = 4$, $m = 0$, (2 1 4 6) describes the projective plane, as does (1 5 2 10 3) for $n = 5$, $m = 0$. But the method does not produce all models, and, indeed, does not even give all models for a given m and n (viz. models A and B).

It is interesting to note that the dual of a plane, as obtained by calling points lines and vice versa, is a collection of points and lines in which there are exactly $n+m$ points on every line, and n lines on every point. No two (distinct) points have more than one line on them, and every two lines have a point in common. There is not a line on every two points. This dual is therefore what is usually called a partial plane of the projective plane, and accordingly can be embedded in a projective plane which has at least $n+m$ points on a line. Exciting possibilities arise if the projective plane of smallest order in which this embedding can occur is a plane of finite order.

Another area of possible research involves the set of all lines parallel to a given line of the plane. This is a miniature geometry in itself, with n points on each line, m lines on each point, and $t-n$ points in all. Unfortunately, there is not necessarily a line on every two points of this "subplane", so it is not a plane as described in this paper.

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