COLLOQUIUM MATHEMATICUM

VOL. XIII 1964 FASC. 1

SOME REMARKS ON MONOTHETIC GROUPS

BY

S. ROLEWICZ (WARSAW)

A topological group G is called *monothetic* if there is an element x such that the group H_x generated by x (i. e. the set of all elements of the type x^n, $n = 0, \pm 1, \pm 2, \ldots$) is dense in G. The idea of monothetic groups was introduced by van Dantzig [1], [2]. These groups have been also investigated by Eckman [3] and by Halmos and Samelson [4], who proved independently, among others, that every separable connected Abelian compact group is monothetic.

If a monothetic group G is locally compact, then it is either discrete or compact (see [5], lemma 1, p. 270).

The following question arises: Is this alternative true under weaker assumptions? Does it hold in the case when G is a complete metric monothetic group?

The answer is negative, as it is shown by the following

THEOREM. *There exists a monothetic complete metric Abelian group G, which is neither discrete nor compact.*

Proof. Let G be the set of sequences of complex numbers $z = \{z_n\}$, where $|z_n| = 1$ and $z_n \to 1$. The group operation is defined as multiplication by coordinates, i. e. the product xy of $x = \{x_n\}$ and $y = \{y_n\}$ is the element $\{x_n y_n\}$.

The *distance* of two elements $x = \{x_n\}$ and $y = \{y_n\}$ is defined by $\varrho(x, y) = \max_n |x_n - y_n|$.

It is easily seen that the group G is complete and non-compact. For any $y = \{y_n\}$ denote by $P_m y$ the element $\{y_1, y_2, \ldots, y_m, 1, 1, \ldots\}$. Let $x = \{x_n\} = \{e^{i\lambda_n}\}$, where $\lambda_1, \lambda_2, \ldots$ are independent over rationals and subject to the condition

$$|\lambda_n| < \frac{1}{2^n k_{n-1}} \quad (n = 2, 3, \ldots),$$

where k_n is the smallest integer k for which the inequality

$$\varrho\big((P_n x)^j, P_n y\big) \leqslant \frac{1}{2^n} \tag{1}$$

can be satisfied for every $y \in G$ and a suitable $j \leqslant k$. The existence of such k's follows from Kronecker's approximation theorem. Of course we have $k_n \leqslant k_{n-1}$.

Let y be an arbitrary element of the group G and ε an arbitrary positive number. It trivially follows from the definition of the group G that there is a number n_0 such that for all $n > n_0$ we have $\varrho(y, P_n y) < \varepsilon/3$. We choose an $n > n_0$ so as to have $1/2^n < \varepsilon/3$. By the manner the sequence $\{x_n\}$ was chosen, there is an integer j $(0 \leqslant j \leqslant k_n)$ such that (1) holds. For $m > n$ one has

$$j\,|\lambda_m| \geqslant \frac{j}{2^n k_n} \geqslant \frac{1}{2^n}.$$

Hence $\varrho\big(x^j, (P_n x)^j\big) < \varepsilon/3$. Therefore

$$\varrho(x^j, y) \leqslant \varrho\big(x^j, (P_n x)^j\big) + \varrho\big((P_n x)^j, P_n y\big) + \varrho(P_n y, y) < \varepsilon$$

and, since y was an arbitrary element of G, the group G is monothetic. This completes the proof.

The author would like to express his thanks to Prof. S. Hartman for his keen remarks, which helped to the preparation of this note.

REFERENCES

[1] D. van Dantzig, *Über topologisch homogene Kontinua*, Fundamenta Mathematicae 15 (1930), p. 102-125.

[2] — *Zur topologischen Algebra I*, Mathematische Annalen 107 (1932), p. 587-626.

[3] B. Eckman, *Über monothetische Gruppen*, Commentàrii Mathematici Helvetici 16 (1943), p. 249-263.

[4] P. R. Halmos and H. Samelson, *On monothetic groups*, Proceedings of the National Academy of Science, USA, 28 (1942), p. 254-258.

[5] А. С. Понтрягин, *Непрерывные группы*, издание второе, Москва 1954.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 12. 9. 1963