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COMMON FIXED POINTS FOR ISOTONE MAPPINGS

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Tarski [4] has obtained various fixed point theorems for isotone mappings of a complete lattice into itself. Unfortunately, his results are not directly applicable in certain interesting situations. For example, Ponomarev [3] has recently obtained a theorem on the existence of a closed non-empty common fixed set for two commuting multivalued mappings in a compact Hausdorff space. Tarski's results are not applicable since the collection of non-empty closed subsets of a compact Hausdorff space do not form a complete lattice except in the trivial case of a space consisting of a single point. In this paper we shall slightly generalize Tarski's results as well as present some new results applicable in many situations, but, in particular, in the case of isotone set-to-set mappings in which only non-empty sets are involved. For a general discussion of partially ordered sets the reader may refer to Birkhoff [1].

Definition 1. Let X be a partially ordered space. An element $c \in X$ is said to be *chain-compact* if, for every non-empty chain $L \subset X$, where $x \leqslant c$ for all $x \in L$, $\inf L$ exists. Recall that a *chain* is a totally ordered subset of X.

This definition is motivated by the so-called "nest characterization of compactness" (see [2], p. 163). It is clear that if c is chain-compact and $c_0 \leqslant c$, then c_0 is chain-compact.

THEOREM 1. *Let X be a partially ordered set and let $\mathscr{I}$ be a non-empty commutative family of isotone mappings of X into itself. If there exists a chain-compact element $c \in X$ such that $f(c) \leqslant c$ for all $f \in \mathscr{I}$, then there exists a minimal common fixed point $a \in X$ for the family $\mathscr{I}$; i. e., $f(a) = a$ for all $f \in \mathscr{I}$ and a is minimal with this property.*

(Recall that a mapping f is *isotone* if $f(x) \leqslant f(y)$ whenever $x \leqslant y$.)

Proof. Define $M = \{x : f(x) \leqslant x \leqslant c \text{ for all } f \in \mathscr{I}\}$. Since $c \in M$, M is non-empty. By Zorn's lemma there exists a maximal chain $L \subset M$. Clearly L is non-empty and since c is chain-compact and $x \leqslant c$ for all $x \in L$, $\inf L = a$ exists. Clearly $f(a) \leqslant f(x) \leqslant x \leqslant c$ for all $x \in L$ and all

$f \in \mathscr{I}$. Hence, $f(a) \leqslant a \leqslant c$ for all $f \in \mathscr{I}$. which means that $a \in M$. Now if $g \in \mathscr{I}$ and we define $b = g(a) \leqslant a$, then

$$f(b) = g\big(f(a)\big) \leqslant g(a) = b \leqslant c$$

for all $f \in \mathscr{I}$. Hence, $b \in M$. Since L is a maximal chain and $b \leqslant a = \inf L$, we must have $b \in L$ which means that $a \leqslant b$; hence, $a = b$. Thus, we have $g(a) = a$ for all $g \in \mathscr{I}$. Now if $x_0 \leqslant a \leqslant c$ and $f(x_0) = x_0$ for all $f \in \mathscr{I}$, then $x_0 \in M$. Since L is a maximal chain and $x_0 \leqslant a = \inf L$, we must have $x_0 \in L$; hence, $x_0 = a$. Thus, a is a minimal common fixed point for the family $\mathscr{I}$, q. e. d.

This result slightly generalizes theorem 2 of Tarski (see [4], p. 288). It can also be applied to generalize a recent result of Ponomarev [3] in the following way: let E be any topological space and let F be a non-empty compact subset of E. Let X be the collection of all (or all closed) subsets of E which have a non-empty closed intersection with F. Let X be partially ordered by inclusion; thus, every element of X is chain-compact. Hence, if $\mathscr{I}$ is any non-empty commutative family of isotone mappings of X into itself, then there exists a minimal non-empty set $A \subset E$ such that $f(A) = A$ for all $f \in \mathscr{I}$.

One may also apply theorem 1 in the following way: let E be a connected compact Hausdorff space and let $\mathscr{I}$ be a commutative family of continuous single-valued mappings of E into itself. If we let X be the collection of all non-empty connected closed subsets of E, then $\mathscr{I}$ can be regarded as a commutative family of isotone mappings of X into itself (for each $f \in \mathscr{I}$ and each $A \in X$ define $f(A) = \{f(x) : x \in A\}$). It is clear that every element of X is chain-compact. Thus, there exists a minimal non-empty connected closed set $A \subset E$ such that $f(A) = A$ for all $f \in \mathscr{I}$.

Definition 2. A partially ordered set X is called a *complete semi-lattice* if, for every non-empty $M \subset X$, $\sup M$ exists.

We note that if X is a complete semi-lattice and if $M \subset X$ is non-empty and bounded below, then $\inf M$ exists.

THEOREM 2. *Let $\mathscr{I}$ be a non-empty commutative family of isotone mappings of a complete semi-lattice X into itself. If there exists $b \in X$ such that*

$$b \leqslant \sup\{f^n(b) : n = 1, 2, \ldots\}$$

for all $f \in \mathscr{I}$, then there exists a common fixed point $a \in X$ for the family $\mathscr{I}$ which is the minimal common fixed point in the set $N = \{x : b \leqslant x\}$.

Proof. Define $M = \{x : b \leqslant x$ and $f(x) \leqslant x$ for all $f \in \mathscr{I}\}$. If we set $e = \sup X$, then $e \in M$ so that M is non-empty. Since M is bounded below, $\inf M = a$ exists. It is clear that $b \leqslant a$ and $f(a) \leqslant a$ for all $f \in \mathscr{I}$;

hence, $a \in M$. Using these facts and the conditions of the theorem, we have

$$b \leqslant \sup\{f^n(b) : n = 1, 2, \ldots\} \leqslant \sup\{f^n(a) : n = 1, 2, \ldots\} = f(a)$$

for all $f \in \mathscr{I}$. Now, for any $g \in \mathscr{I}$, $f(g(a)) = g(f(a)) \leqslant g(a)$ for all $f \in \mathscr{I}$. Hence, $g(a) \in M$ for all $g \in \mathscr{I}$. Thus, $a \leqslant g(a)$ for all $g \in \mathscr{I}$ and therefore $f(a) = a$ for all $f \in \mathscr{I}$. It is clear that a is the minimum common fixed point in the set $N = \{x : b \leqslant x\}$, q. e. d.

LEMMA 3. *If f is an isotone mapping of a complete semi-lattice x into itself and if there exists an element $b \in X$ such that $b \leqslant f(b)$, then f has a maximum fixed point.*

Proof. Define $M = \{x : x \leqslant f(x)\}$ and $a = \sup M$. Thus, $a \leqslant f(a) \leqslant f(f(a))$ so that $f(a) \in M$ and, hence, $f(a) \leqslant a$; i. e., $f(a) = a$. It is clear that if $f(x) = x$, then $x \in M$ and, hence, $x \leqslant a$, Therefore, a is the maximum fixed point, q. e. d.

Note. Although the author proved this lemma independently of Tarski, the original proof is due to Tarski [4]. The proof is given here to make this article self-contained.

THEOREM 4. *Let f and g be commutative isotone mappings of a complete semi-lattice X into itself. If $h(x) = f(g(x))$ has a fixed point, then f and g have a common fixed point.*

Proof. Suppose $b = h(b)$. Since h is an isotone mapping of the complete semi-lattice X into itself, it has a maximum fixed point $a \in X$ (by lemma 3). Since f and g commute with h, $f(a)$ and $g(a)$ are also fixed points for h. Since a is a maximum fixed point for h, we have $f(a) \leqslant a$ and $g(a) \leqslant a$. Therefore, $h(a) = f(g(a)) \leqslant f(a) \leqslant a$ and $h(a) = g(f(a)) \leqslant g(a) \leqslant a$; since $h(a) = a$, we have $f(a) = g(a) = a$, q. e. d.

THEOREM 5. *Let f and g be isotone mappings of a complete semi-lattice X into itself which have the following two properties:*

1. *if $x \leqslant f(x)$, then $f(x) \leqslant g(x)$,* 2. *if $f(x) \leqslant x$, then $g(x) \leqslant f(x)$.*

Then if f or g has a fixed point, there is a common fixed point.

Note. The mappings f and g need not commute. The two properties given above generalize the situation where $g = f^n$ for some integer n.

Proof. Suppose $f(a) = a$. By the two properties given above $g(a) \leqslant a$ and $a \leqslant g(a)$; therefore, $g(a) = a$.

Suppose $g(b) = b$. Define $M = \{x : b \leqslant x \text{ and } f(x) \leqslant x\}$. If we set $e = \sup X$, then $e \in M$ so that M is non-empty. Since M is bounded below, $\inf M = c$ exists. It is clear that $b \leqslant c$ and $f(c) \leqslant c$. From the conditions of the theorem we see that $b = g(b) \leqslant g(c)$ and $g(c) \leqslant f(c)$; hence, $b \leqslant f(c)$ and $f(f(c)) \leqslant f(c)$ so that $f(c) \in M$. This means that $c \leqslant f(c)$; therefore, $f(c) = c = g(c)$, q. e. d.

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