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ON SUBSETS OF INDECOMPOSABLE CONTINUA

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1. Introduction. In this paper are presented some results of a study of compact metric indecomposable continua. Throughout this paper, I denotes the set of all points of a compact metric space and is assumed to be an indecomposable continuum. Theorems 135-142 of [7], Ch. I, are used without further explicit reference to them.

2. Subsets of I which are the sums of countably many closed point sets. We give now the following

Definition. If K is a point set, then the *K-composant* of I, denoted by $CP(K)$, is the set to which p belongs if, and only if, p is a point and there exists a proper subcontinuum of I containing both p and a point of K. Note that, if x is a point, $CP(x)$ is the composant, containing x, of I.

THEOREM 1. *If K is a closed point set, then the K-composant of I is the sum of countably many closed point sets.*

Proof. Suppose that K is a closed point set, x is a point, and $D_1, D_2, \ldots$ is a sequence of domains closing down on x such that $I-D_1$ contains a point of K. For each n, let M_n denote the sum of all the components of $I-D_n$ which intersect K. Suppose that, for some n, p is a limit point of M_n. Then there exists a sequence $p_1, p_2, \ldots$ of points of M_n converging to p. For each i, denote by q_i a point of K lying in the component C_i of $I-D_n$ containing p_i. Then there exists a subsequence $q_{n_1}, q_{n_2}, \ldots$ of the sequence $q_1, q_2, \ldots$ converging to a point q of $K\cdot(I-D_n)$, and the sequence $C_{n_1}, C_{n_2}, \ldots$ has a limiting set which is a continuum C lying in $I-D_n$ and containing both p and q [4]. Hence, p belongs to M_n and M_n is closed. But $CP(K)$ is either $M_1+M_2+\ldots$ or $CP(x)+M_1+M_2+\ldots$ and, in either case, is the sum of countably many closed point sets.

THEOREM 2. *If the point set K is the sum of countably many closed point sets, then so is the K-composant of I.*

Proof. Evidently, if $K = K_1+K_2+\ldots$, then $CP(K) = CP(K_1)+CP(K_2)+\ldots$ Hence, if each K_i is closed, $CP(K)$ is the sum of countably many closed point sets.

THEOREM 3. *Suppose that, for each positive integer n, K_n is a closed subset of I and $K_1+K_2+\ldots$ intersects every composant of I. Then there exists a positive integer n such that K_n intersects every composant of I.*

Proof. Suppose that for no n does K_n intersect every composant of I. For each n, $CP(K_n)$ is the sum of countably many closed point sets such that, if g is one of them, then every point of g is a limit point of $I-g$. But $I = CP(K_1)+CP(K_2)+\ldots$ and, therefore, is the sum of countably many closed point sets such that, if g is one of them, then every point of g is a limit point of $I-g$, contrary to [7], Ch. I, Th. 53.

THEOREM 4. *Suppose that K is the sum of countably many closed point sets (i. e. K is an F_σ-set), and intersects every composant of I. Then K contains a non-empty closed point set M such that, for every composant C of I, $C\cdot M$ is dense in M.*

Proof. By Theorem 3, K contains a closed point set M_1 which intersects every composant of I. Let φ denote the least ordinal whose cardinality is greater than that of the continuum. Denote by Q a well ordered sequence $q_1, q_2, \ldots, q_\alpha, \ldots$ $(\alpha<\varphi)$, such that (1) each term of Q is a composant of I, and (2) if $\alpha<\beta<\varphi$, $q_\alpha = q_\beta$, and C is a composant of I distinct from q_α, then there exists an ordinal γ $(\alpha<\gamma<\beta)$, such that $q_\gamma = C$. Let $M_{11}, M_{12}, \ldots, M_{1\alpha}, \ldots$ $(\alpha<\varphi)$, denote the sequence such that (1) $M_{11} = M_1$; (2) for each $\alpha<\varphi$, $M_{1,\alpha+1} = \mathrm{cl}(q_\alpha\cdot M_{1\alpha})$; and (3) if β is a limit ordinal $<\varphi$, $M_{1\beta}$ is the common part of the closed and compact point sets $M_{11}, M_{12}, \ldots, M_{1\alpha}, \ldots$ $(\alpha<\beta)$. (If H is a point set, $\mathrm{cl}(H)$ denotes the closure of H.)

Suppose that, for some ordinal $\alpha<\varphi$, $M_{1\alpha}$ is empty. Denote the least such ordinal α by α_1. Then α_1 is not a limit ordinal, and q_{α_1-1} does not intersect M_{1,α_1-1}. Now, M_1-M_{1,α_1-1} is the sum of countably many closed point sets $K_{11}, K_{12}, \ldots$; M_{1,α_1-1} is closed; and $M_1 = M_{1,\alpha_1-1}+K_{11}+K_{12}+\ldots$ intersects every composant of I. Thus, by Theorem 3, at least one of the point sets $K_{11}, K_{12}, \ldots$ intersects every composant of I; denote one such by M_2. Let $M_{21}, M_{22}, \ldots, M_{2\alpha}, \ldots$ $(\alpha<\varphi)$ denote the sequence such that (1) $M_{21} = M_2$; (2) for each $\alpha<\varphi$, $M_{2,\alpha+1} = \mathrm{cl}(q_\alpha\cdot M_{2\alpha})$; and (3) if β is a limit ordinal $<\varphi$, M_2 is the common part of the point sets $M_{21}, M_{22}, \ldots, M_{2\alpha}, \ldots$ $(\alpha<\beta)$. Evidently, M_{2,α_1-1} is empty, for, if it were not, it would be a subset of M_{1,α_1-1}, which does not intersect M_2. Let α_2 denote the least ordinal α for which $M_{2\alpha}$ is empty. Then α_2 is not a limit ordinal and $\alpha_2<\alpha_1$. Hence, by repeating the above process, there exists a closed subset M_3 of M_2-M_{2,α_2-1} which intersects every composant of I and an ordinal $a_3 < a_2$. Indeed, by continuing this process, we find that there exists an infinite decreasing sequence $a_1, a_2, \ldots$ of ordinals, which is impossible. Thus, the assumption that there exists an ordinal $a < \varphi$ such that M_{1a} is empty is false.

Now, M_1 is of the power of the continuum. Therefore, there exists an ordinal $a < \varphi$ such that, if δ is an ordinal $(a < \delta < \varphi)$, $M_{1a} = M_{1\delta}$; and, hence, $\mathrm{cl}(q_\delta \cdot M_{1a}) = M_{1a}$. Thus, M_{1a} satisfies the conditions for M in the conclusion of Theorem 4.

3. An application of Theorem 4. Bing [1] has raised the question as to whether there exists a reversibly continuous transformation of a pseudo-arc M into itself, other than the identity, which leaves every composant of M fixed. Since every non-degenerate subcontinuum of a pseudo-arc is a pseudo-arc [6] and has the fixed point property [3], the following theorem may have some bearing on Bing's question:

THEOREM 5. *Suppose that* (1) *every non-degenerate proper subcontinuum of I is indecomposable and has the fixed point property, and* (2) *f is a monotone mapping of I into I such that, for each composant C of I, $f(C) = C$. Then, for every composant C of I, there exist infinitely many points p of C such that $f(p) = p$.*

Proof. Suppose that C is a composant of I such that, if p is a point of C, $f(p) \neq p$. Let x denote a point of C and K_1 denote a proper subcontinuum of I containing both x and $f(x)$. Since $K_1 + f(K_1)$ is an indecomposable continuum, one of the two continua K_1 and $f(K_1)$ is a subcontinuum of the other, and, since K_1 has the fixed point property and f leaves no point of K_1 fixed, K_1 is a proper subset of $f(K_1)$. Similarly, one of the two continua K_1 and $f^{-1}(K_1)$ is a subcontinuum of the other, and, hence, $f^{-1}(K_1)$ is a proper subset of K_1. Let φ denote the least ordinal whose cardinality is greater than that of the continuum. There exists a sequence $K_1, K_2, \ldots, K_a, \ldots$ $(a < \varphi)$, such that (1) if $a < \varphi$, $K_{a+1} = f^{-1}(K_a)$, and (2) if β is a limit ordinal $< \varphi$, K_β is the common part of the compact continua $K_1, K_2, \ldots, K_a, \ldots$ $(a < \beta)$. But, it can be shown by transfinite induction that, if $a < \beta < \varphi$, K_f is a proper subcontinuum of K_a, which is contrary to the fact that K_1 is of the power of the continuum. Thus, for each composant C of I, there exists a point p of C such that $f(p) = p$. Then, M, the set of all points p of I such that $f(p) = p$, intersects every composant of I. Therefore, since M is closed, it follows from Theorem 4 that, if C is a composant of I, $C \cdot M$ is infinite.

4. Closed and totally disconnected subsets of I. Mazurkiewicz [5] proved that I contains a perfect point set M such that no composant of I contains two points of M. Theorem 8 generalizes this result and answers a question asked the author (in conversation) by Professor B. J. Ball.

The statement that a point p of a subset K of a continuum M is continuumwise accessible from $M-K$ means that there is a non-degenerate continuum that contains p and lies wholly in $(M-K)+p$, [2]. Cornette [2] was the first to show the existence of a compact metric continuum M containing a totally disconnected perfect point set K no point of which is continuumwise accessible from $M-K$. His examples were hereditarily decomposable chainable continua but his proofs suggested the proof of Theorem 10.

THEOREM 6. *Suppose that, in a metric space, S is the set of all points and is a compact continuum. In order that S should be an irreducible continuum from the point x to the point y it is necessary and sufficient that, for every finite collection G of domains covering S, there should exist a positive number ε such that if D is an ε-chain from x to y, then each domain of the collection G intersects some link of D.*

Proof. Suppose that M is a proper subcontinuum of S containing x and y, p is a point of $S-M$, δ is the distance from p to M, and G is a finite collection of domains, covering S, each having diameter less than $\delta/2$. For each positive number $\varepsilon<\delta/2$ there exists ([7], Ch. I, Th. 165) an ε-chain from x to y, each link of which intersects M and, hence, no link of which intersects a domain of the collection G which contains p.

Suppose, on the other hand, that G is a finite collection of domains covering S and, for each n, D_n is a $1/n$-chain from x to y such that some domain of the collection G does not intersect any link of D_n. There is a domain g of G and an infinite subsequence $D_{n_1}, D_{n_2}, \ldots$ of the sequence $D_1, D_2, \ldots$ such that no link of any term of that subsequence intersects g. Then the limiting set of the sequence $D^*_{n_1}, D^*_{n_2}, \ldots$ of point sets is a continuum containing x and y, [4], but no point of g.

THEOREM 7. *Suppose that K is a finite subset of I such that no composant of I contains two points of K, and G is a finite collection of domains covering I. Then there exists a positive number ε such that, if D is an ε-chain from one point of K to another, then every domain of G intersects some link of D.*

THEOREM 8. *Suppose that M is an inner limiting subset of I (i. e. M is a G_δ-set) which intersects uncountably many composants of I. Then M contains a perfect point set S such that no proper subcontinuum of I contains two points of C, but no such point set C intersects every composant of I.*

Proof. For each composant K of I which intersects M, denote by p_K a particular point of $M\cdot K$. Let M' denote the set of all points p_K for all composants K of I which intersect M, and let M'' denote the set to which x belongs if, and only if, x is a point of M' such that every domain which contains x contains uncountably many points of M'. Every domain which contains a point of M'' contains uncountably many points of M''. Let $E_1, E_2, \ldots$ denote a sequence of domains whose common part is M. Let a and b denote two points of M'' and let Z_1 denote the set whose only elements are a and b. Let G_1 denote a finite collection of domains, each of diameter less than 1, covering I and containing two domains g_a and g_b, containing a and b respectively, whose closures are mutually exclusive subsets of E_1. Let H_1 denote the collection whose only elements are g_a and g_b. Denote by ε_1 a positive number less than 1/2 such that if D is an ε_1-chain from a to b, then every domain of G_1 intersects some link of D. Let $Z_1, Z_2, \ldots$; $G_1, G_2, \ldots$; $H_1, H_2, \ldots$; and $\varepsilon_1, \varepsilon_2, \ldots$ denote sequences such that, for each n, (1) Z_{n+1} is a set of only 2^{n+1} points of M'' such that each domain h of H_n contains only two of them, each at a distance less than $\varepsilon_n/3$ from the point of $h \cdot Z_n$; (2) G_{n+1} is a finite collection of domains covering I, each having diameter less than ε_n, such that the closure of each domain of G_{n+1} which contains a point of Z_{n+1} is a subset of E_{n+1} and of some domain of H_n and intersects the closure of no other domain of G_{n+1} intersecting Z_{n+1}; (3) H_{n+1} is the collection consisting only of the 2^{n+1} domains of G_{n+1} which intersect Z_{n+1}; and (4) ε_{n+1} is a positive number less than $1/(n+2)$ such that if D is an ε_{n+1}-chain from one point of Z_{n+1} to another, then every domain of G_{n+1} intersects some link of D.

Denote by C the common part of the point sets $H_1^*, H_2^*, \ldots$ Evidently, C is a totally disconnected perfect point set and is a subset of M. Suppose that x and y are two points of C, T is a proper subcontinuum of I containing x and y, and p is a point of $I-T$. Let $n > 1$ denote an integer such that the distance from p to T is greater than $\varepsilon_{n-1}+2\varepsilon_n$ and the distance from x to y is greater than ε_{n-1}. There exist an ε_n-chain D from x to y, each link of which intersects T and points x' and y' of Z_n such that x belongs to the domain of H_n containing x' and y belongs to the domain of H_n containing y'. Let R_x denote the set of all points of I at a distance less than $\varepsilon_n/3$ from x' and let R_y denote the set of all points of I at a distance less than $\varepsilon_n/3$ from y'. Then there exists a chain D' whose first link is R_x, whose last link is R_y, and each of whose links distinct from R_x and R_y is either a link of D or a domain of H_{n+1} containing either x or y. But D' is an ε_n-chain from x' to y' no one of whose links intersects a domain of G_n which contains p, contrary to the stipulation (4) above. Therefore, no proper subcontinuum of I contains two points of C. That no such point set C intersects every composant of I follows from Theorem 4.

COROLLARY. *If M is an inner limiting subset of I, then the collection of all composants of I with intersect M is either countable or of the power of the continuum* (cf. [5]).

THEOREM 9. *If M is an inner limiting subset of I which contains a composant of I, then the collection of all composants of I which lie wholly in M is of the power of the continuum.*

Proof. Let G be the collection of all composants of I which lie wholly in M. Evidently $CP(I-M) = I-G^*$, hence, by Theorem 2, G^* is an inner limiting set, and, by the corollary to Theorem 8, G is either countable or of the power of the continuum. Suppose that G is countable; $g_1, g_2, \ldots$ are the elements of G; and, for each n, p_n is a point of g_n. Now, $I-M$ is the sum of countably many closed point sets $K_1, K_2, \ldots$ But then $(p_1+p_2+\ldots)+K_1+K_2+\ldots$ intersects every composant of I and is the sum of countably many closed point sets no one of which intersects every composant of I, contrary to Theorem 3. Hence, G is of the power of the continuum.

THEOREM 10. *If I is hereditarily indecomposable, there exists a totally disconnected perfect point set C such that no point of C is continuumwise accessible from $I-C$.*

Proof. Denote by $K_1, K_2, \ldots$ a monotonic sequence of closed subsets of I such that, for each n, K_n has only 2^n components, each having diameter less than $1/n$, and each containing two components of K_{n+1}. Evidently, the common part C of the point sets of that sequence is a totally disconnected perfect point set. Suppose that M is a non-degenerate subcontinuum of I containing only one point p of C, ε is the diameter of M, n is a positive integer greater than $1/\varepsilon$, and k is the component of K_n containing p. Then $M+k$ is an indecomposable continuum and k has diameter less than that of M. Then k is a subset of M. But k contains uncountably many points of C.

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