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ON THE DUAL SPACE OF $H_B^{1,\infty}$

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1. Introduction. When we are dealing with Hardy space $H_B^p(D)$ of B-valued analytic functions on the disk D for some p $(1 \leqslant p \leqslant \infty)$, and we want to obtain the functions in $L_B^p(T)$ with $\hat{f}(n) = 0$ for $n < 0$ as boundary values of this space, we have to require a certain property on B. This property was defined by Bukhvalov and Danilevich [4] and it was called the *analytic Radon–Nikodym property*.

Throughout the paper we are concerned with Hardy spaces defined on the boundary of D and some questions about duality will be studied. Some results about this subject were considered in [3] for $1 < p < \infty$ and we will study here the case $p = 1$.

We denote by H_B^1 the space of Bochner-integrable functions f in $L^1(T)$ such that $\hat{f}(n) = 0$ for $n < 0$, and by $H_B^{1,\infty}$ the space defined below in terms of B-valued atoms. Bourgain has recently proved [2] that every function f in H_B^1 can be decomposed into B-atoms, i.e., $H_B^1 \subset H_B^{1,\infty}$. We actually know that both spaces coincide if and only if B has the U.M.D. property ([1], [2]).

We are interested in obtaining a representation of $(H_B^{1,\infty})^*$.

First of all we recall what happens in the scalar case. It is well known that the space of functions of bounded mean oscillation (BMO), defined by John and Nirenberg [8], may be viewed as the dual space of $\operatorname{Re} H^1$. This last result was proved by Fefferman [7]. Subsequently, R. Coifman showed that $\operatorname{Re} H^1$ could be defined by atoms, i.e., $H^1 = H^{1,\infty}$, and a direct proof of the duality $(H^{1,\infty})^* = \mathrm{BMO}$ may be found in [5].

On the other hand, let us recall that when we take functions with values in a Banach space B, and we intend to give a representation of the dual space of $L_B^p(T)$, the geometry on the space B must be considered. In fact, for $1 \leqslant p < \infty$,

(1.1) $(L_B^p)^* = L_{B^*}^{p'}$ *if and only if* B^* *has the R.N.P.* ([6]).

Both facts suggest the following result which will be proved in this paper:

(1.2) $(H_B^{1,\infty})^* = \mathrm{BMO}_{B^*}$ *if and only if* B^* *has the R.N.P.*

2. Definitions and lemma. Let $1 < p \leqslant \infty$ and let $a \in L_B^p$. We say that a is a $(1, p, B)$-*atom* if

(1) $\operatorname{supp} a \subset I$, I is an interval of T;

(2) $\|a\|_p \leqslant 1/m(I)^{1/q}$, $1/p + 1/q = 1$ (m is Lebesgue measure);

(3) $\int_I a(t)\,dt = 0$.

The function $a(t) = b\chi_T(t)$, where $\|b\|_B = 1$, is also considered a $(1, p, B)$-atom (χ_E denotes the characteristic function of E). We define (see [5])

$$H_B^{1,p} = \Big\{f \in L_B^1 \,\Big|\, f(t) = \sum_{i=1}^{\infty} \lambda_i a_i(t),\ \sum_{i=1}^{\infty} |\lambda_i| < \infty \text{ and the } a_i\text{'s are } (1, p, B)\text{-atoms}\Big\},$$

and if we put

$$\|f\|_{H_B^{1,p}} = \inf \sum_{i=1}^{\infty} |\lambda_i|,$$

where the infimum is taken over all the representations of f, then $(H_B^{1,p}, \|\ \|_{H_B^{1,p}})$ is a Banach space. It is easy to see that

(2.1) *If f belongs to $H_B^{1,p}$ and*

$$f = \sum_{i=1}^{\infty} \lambda_i a_i,$$

then $\sum_{i=1}^{N} \lambda_i a_i$ converges to f in $H_B^{1,p}$ when $N \to \infty$.

Let $1 \leqslant q < \infty$: we define (see [5])

$$\mathrm{BMO}_B^q = \left\{f \in L_B^q \,\Big|\, \sup_I \left(\frac{1}{m(I)} \int_I \|f(t) - f_I\|_B^q\,dt\right)^{1/q} < \infty\right\},$$

where I denotes an interval and

$$f_I = \frac{1}{m(I)} \int_I f(t)\,dt.$$

If we put

$$\|f\|_{\mathrm{BMO}_B^q} = \eta_{q,B}(f) + \Big\|\int_T f(t)\,dt\Big\|_B,$$

where

$$\eta_{q,B}(f) = \inf\left\{C:\ \sup_I \left(\frac{1}{m(I)} \int_I \|f(t) - f_I\|_B^q\,dt\right)^{1/q} \leqslant C\right\},$$

then $(\mathrm{BMO}_B^q, \|\ \|_{\mathrm{BMO}_B^q})$ is a Banach space for every q $(1 \leqslant q < \infty)$. We have just defined BMO_B^q for different values of q, but we actually have

(2.2) *For every* q $(1 < q < \infty)$,

$$\mathrm{BMO}_B^q = \mathrm{BMO}_B^1 \quad \textit{and} \quad \|\ \|_{\mathrm{BMO}_B^q} \sim \|\ \|_{\mathrm{BMO}_B^1}.$$

The proof of (2.2) is a corollary to John and Nirenberg's lemma [8] since the technique may be reproduced by merely changing the absolute value by the norm in B.

LEMMA. *If* $1 < p \leqslant \infty$, *then* $L_B^p \subset H_B^{1,p} \subset L_B^1$ *and the embeddings are continuous.*

Proof. Given $f \in L_B^p$, f may be written in the following way:

$$f = \Big\|\int_T f(t)\,dt\Big\|_B a_1(t) + 2\|f\|_p a_2(t),$$

where

$$a_1(t) = \frac{\int_T f(t)\,dt}{\Big\|\int_T f(t)\,dt\Big\|}\chi_T(t) \quad \text{and} \quad a_2(t) = \frac{f(t) - \int_T f(s)\,ds}{2\|f\|_p}$$

are clearly $(1, p, B)$-atoms. Moreover,

$$\|f\|_{H_B^{1,p}} \leqslant \Big\|\int_T f(t)\,dt\Big\| + 2\|f\|_p \leqslant 3\|f\|_p.$$

For the second embedding, let $1 < p < \infty$ and let a be a $(1, p, B)$-atom. Due to Hölder's inequality and the definition of $(1, p, B)$-atom we have

$$\text{(2.3)} \quad \int_T \|a(t)\|_B\,dt = \int_I \|a(t)\|_B\,dt \leqslant \|a\|_p \Big(\int_T |\chi_I(t)|^q\Big)^{1/q} \leqslant \frac{1}{m(I)^{1/q}} m(I)^{1/q} = 1.$$

(The case $p = \infty$ is easier.)

By (2.3), if f belongs to $H_B^{1,p}$ and

$$f = \sum_{i=1}^{\infty} \lambda_i a_i,$$

then

$$\|f\|_1 \leqslant \sum_{i=1}^{\infty} |\lambda_i|,$$

and so $\|f\|_1 \leqslant \|f\|_{H_B^{1,p}}$.

3. Theorem.

THEOREM. (a) *If* $1 < p \leqslant \infty$ *and* $1/p + 1/q = 1$, *then*

$$\mathrm{BMO}_B^q \subset (H_B^{1,p})^*.$$

(b) *If* $1 < p < \infty$, $1/p + 1/q = 1$ *and* B^* *has the Radon–Nikodym property, then*

$$(H_B^{1,p})^* \subset \mathrm{BMO}_{B^*}^q.$$

(c) *If there exists a number* p $(1 < p \leqslant \infty)$ *such that* $(H_B^{1,p})^* = \mathrm{BMO}_{B^*}^q$, *then* B^* *has the Radon–Nikodym property.*

Proof. (a) Let $1 < p \leqslant \infty$ and let g be a function in $\mathrm{BMO}_{B^*}^q$. We define T_g: $H_B^{1,p} \to R$ in the following way: Let a be a $(1, p, B)$-atom such that

$$\int_I a(t)\,dt = 0;$$

then

$$T_g(a) = \int_T \langle g(t), a(t)\rangle\,dt, \tag{3.1}$$

where $\langle\, ,\rangle$ denotes the duality between B and B^*.

Since a belongs to L_B^p and g belongs to $\mathrm{BMO}_{B^*}^q \subset L_{B^*}^q$, (3.1) is well defined.

It is immediate to show that if g belongs to $L_{B^*}^q$, φ belongs to L_B^p, and J is an interval:

$$\int_J \langle g(t), \varphi(t) - \varphi_J\rangle\,dt = \int_J \langle g(t) - g_J, \varphi(t)\rangle\,dt. \tag{3.2}$$

Using (3.2), Hölder's inequality and

$$\int_I a(s)\,ds = 0,$$

we obtain

$$|T_g(a)| \leqslant \Big(\int_I \|g(t) - g_I\|_{B^*}^q\,dt\Big)^{1/q} \|a\|_p$$

$$\leqslant \left(\frac{1}{m(I)} \int_I \|g(t) - g_I\|_{B^*}^q\right)^{1/q} \leqslant \|g\|_{\mathrm{BMO}_{B^*}^q}.$$

For an atom of the form $a = b\chi_T$ we have

$$|T_g(a)| \leqslant \|b\|_B \Big\|\int_T g(t)\,dt\Big\|_{B^*} \leqslant \|g\|_{\mathrm{BMO}_{B^*}^q}.$$

Now an argument like in [5], p. 632, leads us to considering T_g in $(H_B^{1,p})^*$ and $\|T_g\| \leqslant \|g\|_{\mathrm{BMO}_{B^*}^q}$.

(b) Let $1 < p < \infty$ and let T be an element of $(H_B^{1,p})^*$. By the Lemma, for every $\varphi \in L_B^p$ we obtain

$$|T(\varphi)| \leqslant \|T\| \cdot \|\varphi\|_{H_B^{1,p}} \leqslant 3\|T\| \cdot \|\varphi\|_{L_B^p}. \tag{3.3}$$

Then T may be considered as an element of $(L_B^p)^*$ and since B^* has the Radon–Nikodym property, (1.1) implies that there exists a function g in $L_{B^*}^q$ such that

$$T(\varphi) = \int_T \langle g(t), \varphi(t)\rangle dt \quad \text{for every } \varphi \in L_B^p.$$

We have to prove that g belongs to BMO_{B^*}. First of all,

$$\begin{aligned} \Big\|\int_T g(t)\,dt\Big\|_{B^*} &= \sup_{\|b\|_B = 1} \Big|\int \langle b\chi_T(t), g(t)\rangle dt\Big| \\ &= \sup_{\|b\|_B = 1} |T(b\chi_T)| \leqslant \|T\|. \end{aligned} \tag{3.4}$$

Let I be an interval. By (1.1) and (3.2) we have

$$\begin{aligned} \left(\int_T \left\|\frac{g(t)-g_I}{m(I)^{1/q}}\right\|_{B^*}^q dt\right)^{1/q} &= \sup\left\{\left|\int_I \left\langle \frac{g(t)-g_I}{m(I)^{1/q}}, \varphi(t)\right\rangle dt\right|, \ \|\varphi\|_{L_B^p(I)} \leqslant 1\right\} \\ &= \sup\left\{\left|\int_I \left\langle g(t), \frac{\varphi(t)-\varphi_I}{m(I)^{1/q}}\right\rangle dt\right|, \ \|\varphi\|_{L_B^p(I)} \leqslant 1\right\} \\ &= 2\sup\left\{\left|T\left(\frac{\varphi-\varphi_I}{2m(I)^{1/q}}\chi_I\right)\right|, \ \|\varphi\|_{L_B^p(I)} \leqslant 1\right\} \\ &\leqslant 2\sup\{|T(\psi)|, \ \|\psi\|_{H_B^{1,p}} \leqslant 1\} = 2\|T\|. \end{aligned}$$

Using this together with (3.4), we get

$$\|g\|_{\mathrm{BMO}_{B^*}^q} \leqslant 3\|T\|.$$

(c) In order to show that B^* has the Radon–Nikodym property we are going to prove the following equivalent result (see [6], p. 63):

(3.5) *For every T in $L(L^1, B^*)$ there is a function g in $L_{B^*}^\infty$ such that*

$$T(\alpha) = \int_T \alpha(t) g(t)\,dt \quad \textit{for every } \alpha \textit{ in } L^1.$$

We fix an operator T in $L(L^1, B^*)$ and define $\tilde{T}\colon L_B^1 \to R$ by

$$\tilde{T}\Big(\sum_{i=1}^n b_i \chi_{E_i}\Big) = \sum_{i=1}^n \langle T(\chi_{E_i}), b_i\rangle,$$

where b_i belong to B and $\{E_i\}$ are disjoint measurable sets. It is obvious that

$$|\tilde{T}(\sum_{i=1}^{n} b_i \chi_{E_i})| \leqslant \sum_{i=1}^{n} \|b_i\|_B \|T\| m(E_i) = \|T\| \cdot \|\sum_{i=1}^{n} b_i \chi_{E_i}\|_{L_B^1}.$$

By density, T is extended to $(L_B^1)^*$. Using the value p in the hypothesis and the Lemma, we obtain

$$|\tilde{T}(\varphi)| \leqslant \|T\| \cdot \|\varphi\|_{H_B^{1,p}} \quad \text{for every } \varphi \text{ in } H_B^{1,p},$$

and again $\tilde{T}$ may be considered as an element of $(H_B^{1,p})^*$. Therefore, there is a g in $\mathrm{BMO}_{B^*}^q$ such that

$$\tilde{T}(\varphi) = \int_{\mathbf{T}} \langle g(t), \varphi(t)\rangle dt \quad \text{for every } \varphi \text{ in } H_B^{1,p}.$$

We have only to prove that g is bounded almost everywhere. Since g belongs to $L_{B^*}^1$, putting $I_\varepsilon(t) = (t-\varepsilon, t+\varepsilon)$ we have

$$\begin{aligned} \|\int_{I_\varepsilon(t)} g(s)\, ds\|_{B^*} &= \sup_{\|b\|_B = 1} |\int_{I_\varepsilon(t)} \langle b, g(s)\rangle ds| \\ &= \sup_{\|b\|_B = 1} |\tilde{T}(b\chi_{I_\varepsilon(t)})| = \sup_{\|b\| = 1} |\langle b, T(\chi_{I_\varepsilon(t)})\rangle| \\ &= \|T(\chi_{I_\varepsilon(t)})\|_{B^*} \leqslant \|T\| m(I_\varepsilon) = \|T\| \cdot 2\varepsilon. \end{aligned}$$

Using Lebesgue's differentiation theorem, we have

$$g(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{I_\varepsilon(t)} g(s)\, ds \text{ a.e.,}$$

and so $\|g(t)\|_{B^*} \leqslant \|T\|$ a.e.

COROLLARY. (a) *If B^* has the Radon–Nikodym property and $1 < p < \infty$, then $H_B^{1,p} = H_B^{1,\infty}$ with equivalent norms.*

(b) $(H_B^{1,\infty})^* = \mathrm{BMO}_{B^*}^1$ *if and only if B^* has the Radon–Nikodym property.*

Proof. Given $1 < p < \infty$, let a be a $(1, \infty, B)$-atom. It is clear that

$$\|a\|_p = \left(\int_I \|a(t)\|^p dt\right)^{1/p} \leqslant \|a\|_\infty m(I)^{1/p} \leqslant \frac{1}{m(I)^{1/q}}.$$

Consequently, $H_B^{1,\infty} \subset H_B^{1,p}$, and if f belongs to $H_B^{1,\infty}$, then

$$\|f\|_{H_B^{1,p}} \leqslant \|f\|_{H_B^{1,\infty}}.$$

Now, using part (b) of the Theorem we have

$$(H_B^{1,\infty})^* = \mathrm{BMO}_{B^*}^1 \quad \text{and} \quad (H_B^{1,p})^* = \mathrm{BMO}_{B^*}^q. \tag{3.6}$$

Because of (2.2) and the representation of the dual spaces in (3.6), we obtain part (a). Now, part (b) is an immediate consequence of the Theorem.

Remark. Since C has the Radon–Nikodym property we have just proved that $H_C^{1,p} = H_C^{1,\infty}$, which can be found in [5]. But, on the other hand, the condition on B^* is not necessary in the latter corollary since it may be proved as in [5].

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