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PROPER CONGRUENCES DO NOT IMPLY A MODULAR CONGRUENCE LATTICE

BY

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1. Introduction. If $\mathfrak{A} = \langle A, f_i \rangle_{i \in I}$ is a universal algebra, by a *congruence* R on $\mathfrak{A}$ we mean an equivalence relation $R \subset A \times A$ which is preserved by the operations of $\mathfrak{A}$; that is, for each n-ary operation f_j and $a_i R b_i$ $i = 0, 1, \ldots, n-1$, we have $f_j(a_0 \ldots a_{n-1}) R f_j(b_0 \ldots b_{n-1})$. The congruences of $\mathfrak{A}$, $\mathrm{Co}\,\mathfrak{A}$, form a lattice under set inclusion. If $\mathrm{Co}\,\mathfrak{A}$ is modular, this fact has important bearing on the determination of structural properties of $\mathfrak{A}$. Permutability of congruences is known to imply modularity of the congruence lattice ([1], p. 162). The example given in Section 3 shows that proper congruences (see Section 2) do not imply modularity of the congruence lattice.

2. Background. The congruences of an algebra $\mathfrak{A}$ are said to be *permutable* provided the relative product $R|S = S|R$ for all $R, S \in \mathrm{Co}\,\mathfrak{A}$. $\mathfrak{A}$ is said to have *proper congruences* provided no distinct $R, S \in \mathrm{Co}\,\mathfrak{A}$ have a common equivalence class. Groups for example have both permutable and proper congruences, whereas semigroups in general have neither permutable nor proper congruences. In algebras with proper congruences one may in principle replace the study of congruences by a study of Z-ideals, the equivalence classes of any fixed element Z. Various authors have investigated relationships among the notions proper, permutable, and modular $\mathrm{Co}\,\mathfrak{A}$. Mal'cev[2] has shown permutable congruences do not imply proper congruences for a given algebra $\mathfrak{A}$. Conversely, Valuce[4] has shown proper congruences do not imply permutable congruences. Several easy examples (e. g.: the trivial semigroup on three elements) show that modularity of $\mathrm{Co}\,\mathfrak{A}$ implies neither proper nor permutable congruences. The counterexample of this paper settles in the negative the only remaining possible implication among these three notions.

3. Construction of $\mathfrak{A}$ and $\mathrm{Co}\,\mathfrak{A}$. We desire to construct an algebra $\mathfrak{A} = \langle A, f_i \rangle_{i \in I}$, with proper congruences, for which $\mathrm{Co}\,\mathfrak{A}$ is non-modular. The algebra will be multi-unary, and the congruence lattice will be the familiar 5 element non-modular lattice. Let $A = 10$. In addition to Id = identity relation on A, and $A \times A$ = universal relation on A, we consider equivalence relations defined by partitions of A as follows: $R_0 = \{\{0, 1, 2\}, \{3, 4\}, \{5, 6, 7\}, \{8, 9\}\}$; $R_1 = \{\{0, 1, 2, 3, 4\} \{5, 6, 7, 8, 9\}\}$; $R_2 = \{\{0, 7\}, \{1, 9\}, \{2, 5\}, \{3, 8\}, \{4, 6\}\}$. The task is now to introduce operations on A which will cut down the lattice $\mathrm{Co}\,\mathfrak{A}$ to precisely Id, $A \times A$, R_0, R_1, R_2.

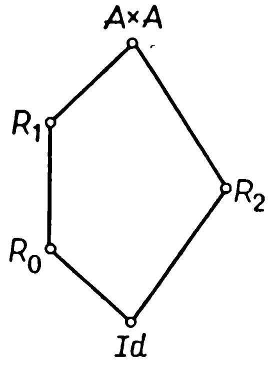

Fig. 1

We will then be done, since these relations will be *proper congruences*, and the lattice $\mathrm{Co}\,\mathfrak{A}$ will have the form in Fig. 1. The algebra in question will be $\mathfrak{A} = \langle A, \varphi_{abcd}, \psi_{ab}, \lambda\rangle$, $(a, b) \in R_2 - Id$, $(c, d) \in R_0 - Id$ where the operations are all unary, defined as follows:

$$\varphi_{abcd}(x) = \begin{cases} c & \text{if } x = a \text{ or } b \\ d & \text{otherwise;} \end{cases} \qquad \psi_{ab}(x) = \begin{cases} a & \text{if } xR_1a \\ b & \text{if } xR_1b \end{cases}$$

$\lambda(x) = x + 5 \pmod{10}$.

Note each of these operations is well defined, and indeed each preserves the relations Id, $A \times A$, R_0, R_1, R_2, as one can easily check.

4. Verification of structure of $\mathrm{Co}\,\mathfrak{A}$. It is now a matter of seeing that these operations preserve no other equivalence relations on A. We proceed by several lemmas. Let $R \subseteq A \times A$ be an equivalence relation, $R \neq Id$, $A \times A$, R_0, R_1, R_2. We shall think of R as being given by a partition of A.

LEMMA 1. *If $(a, b) \in R \in \mathrm{Co}\,\mathfrak{A}$, and $(a, b) \notin R_2$ then $R_0 \subseteq R$.*

Proof. Let $\{a, f\}$ be the R_2 class of a, and let $(c, d) \in R_0$. If $c = d$ then $(c, d) \in R$. If $c \neq d$ then φ_{afcd} is among our opearations. Now $(a, b) \in R \Rightarrow (\varphi_{afcd}(a), \varphi_{afcd}(b)) \in R$, and $\varphi_{afcd}(a) = c$. Also $(a, b) \notin R_2 \Rightarrow \varphi_{afcd}(b) = d$, therefore $(c, d) \in R$ and $R_0 \subseteq R$ as asserted.

LEMMA 2. *If $R \in \mathrm{Co}\,\mathfrak{A}$ has any equivalence class with at least three elements, then $R_0 \subseteq R$.*

Proof. Immediate from Lemma 1 since each R_2 class has only two elements.

LEMMA 3. *If $R \in \mathrm{Co}\,\mathfrak{A}$ has precisely two elements in all of its equivalence classes then $R = R_2$.*

Proof. If $R \neq R_2$ then $\exists (a, b) \in R - R_2$ and by lemma 1 $R_0 \subseteq R$, but this is a contradiction since R_0 has some three element equivalence classes. Now we infer from lemmas 1, 2 and 3 that for $R \in \mathrm{Co}\,\mathfrak{A}$, if $R \neq Id$, $A \times A$, R_0, R_1, R_2 then either $R_0 \subset R$ or $R \subset R_2$. We consider these two cases separately.

Case 1. $R \subset R_2$. From the fact $R \neq Id$, we can fix $(c, d) \in R - Id$, thus also $(c, d) \in R_2$ and $(c, d) \notin R_1$. Since $R \neq R_2$ we can also fix $(a, b) \in R_2 - R$ and thus $(a, b) \neq (c, d)$. Further (as one can easily see from the construction of R_2, i. e.: "taking one small and one large element") we can say without loss of generality $(a, c) \in R_1$ and $(b, d) \in R_1$. Now ψ_{ab} is one of the operations of $\mathfrak{A}$ and $(c, d) \in R$, whereas $\psi_{ab}(c) = a$, $\psi_{ab}(d) = b$ and $(a, b) \notin R$; thus $R \notin \mathrm{Co}\,\mathfrak{A}$.

Case 2. $R_0 \subset R$. *Sub-case 1.* $R \subset R_1$ (and of course $R_0 \neq R \neq R_1$). Then there are only two possibilities for R, namely: $R_3 = \{\{0, 1, 2, 3, 4\}, \{5, 6, 7\}, \{8, 9\}\}$ or $R_4 = \{\{0, 1, 2\}, \{3, 4\}, \{5, 6, 7, 8, 9\}\}$. In both cases λ fails to preserve R: $(2, 3) \in R_3$ but $\lambda(2) = 7$, $\lambda(3) = 8$ and $(7, 8) \notin R_3$; $(7, 8) \in R_4$ but $\lambda(7) = 2$, $\lambda(8) = 3$ and $(2, 3) \notin R_4$.

Sub-case 2. $R \not\subset R_1$. Say $(c, d) \in R - R_1$. Note that $R_2 \not\subset R$ since the transitive closure of R_0 and R_2 is $A \times A$, and thus $R_0 \subseteq R$, $R_2 \subseteq R \Rightarrow R = A \times A$. Then let $(a, b) \in R_2 - R$. Now we may say without loss of generality that $(a, c) \in R_1$ and $(b, d) \in R_1$ (a and b are in separate R_1 classes, as are c and d). ψ_{ab} is an operation of $\mathfrak{A}$, $(c, d) \in R$ and $\psi_{ab}(c) = a$, $\psi_{ab}(d) = b$ but $(a, b) \notin R$ so $R \notin \mathrm{Co}\,\mathfrak{A}$.

This concludes the proof that $\mathrm{Co}\,\mathfrak{A}$ consists precisely of Id, $A \times A$, R_0, R_1, R_2. The example here provides another proof[4] that proper congruences do not imply permutable congruences; this is easy to see directly since $(3, 6) \in R_0 | R_2$ whereas $(3, 6) \notin R_2 | R_0$.

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