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TOTALLY DISCONTINUOUS CONNECTIVITY FUNCTIONS

BY

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Definitions. Suppose f is a function from a topological space X into a topological space Y. The statement that f is a *connectivity function* means that if C is a connected subset of X, then $\{(x, f(x)) | x \text{ is in } C\}$ is a connected subset of $X \times Y$. The statement that f is *peripherally continuous* means that if U is an open subset of X containing a point x of X and V is an open subset of Y containing $f(x)$, then there is an open subset W of U containing x such that $f[B(W)]$ is a subset of V, where $B(W)$ is the boundary of W. The statement that f is *almost continuous* means that if D is an open subset of $X \times Y$ containing the graph of f, then there is a continuous function g from X into Y which has a graph lying in D. The statement that f is *dense in* $X \times Y$ means that if U and V are open subsets of X and Y, respectively, there is a point x of U such that $f(x)$ is in V. Notice that if Y is a non-degenerate Hausdorff space and f is dense in $X \times Y$, then f is totally discontinuous (i.e. nowhere continuous).

Introduction. Many examples of totally discontinuous connectivity functions of real variable have appeared. In fact, Cornette has shown [3] that if Y is a connected separable metric space, there is a connectivity function with domain the unit interval I and range Y, and the function constructed in that argument is dense in $I \times Y$. The techniques used in the construction of these examples rely on the axiom of choice and the fact that in order that a function with a connected real domain be a connectivity function, it is necessary and sufficient that the graph of the entire function be connected. On the other hand, if f is a function from I^2 into the numbers, having the entire graph of f be connected is not sufficient to insure that f be a connectivity function. Real valued connectivity functions with domain I^2 are in some ways better behaved than those with domain I. For example, it follows from Corollary 1 of [9] that every real valued connectivity function with domain I^2 is almost continuous, whereas examples have been given [3], [6], [8] of real connectivity functions which are not almost continuous. Although examples of connectivity functions with domain I^2 having finitely many discontinuities have appeared, [5], [7], a totally discontinuous connectivity function with domain I^2 has not been presented. A connectivity function from I^2 into I will be described here which is not only totally discontinuous, but is dense in $I^2 \times I$. An effective example is constructed using an extension of the technique used for real functions in [2].

The Example. It follows from theorems in [4], [5], [9] and [10] that a function from I^2 into I is a connectivity function if and only if it is peripherally continuous.

Let D denote the square disc $\{(x, y) | \max(|x|, |y|) \leqslant 1\}$. Let A denote the "middle third" Cantor subset of I, and let M' be the set $\{(x, y) | \max(|x|, |y|)$ is in $A\}$. Now, countably many horizontal segments and countably vertical segments, all with endpoints on M', will be added to M'. Notice that if the segment (a, b) is a component of $I - A$, then $b/(b-a)$ is an integer greater than 1.

Let H be the collection such that S belongs to H if and only if there is a component (a, b) of $I - A$ and an integer n such that $|n| < b/(b-a)$ and S is the horizontal segment with end points $\big(a, n(b-a)\big)$ and $\big(b, n(b-a)\big)$ or S is the horizontal segment with endpoints $\big(-a, n(b-a)\big)$ and $\big(-b, n(b-a)\big)$. Let V be the collection such that S belongs to V if and only if there is a component (a, b) of $I - A$ and an integer n such that $|n| < b/(b-a)$ and S is the vertical segment with endpoints $\big(n(b-a), a\big)$ and $\big(n(b-a), b\big)$ or S is the vertical segment with endpoints $\big(n(b-a), -b\big)$ and $\big(n(b-a), -a\big)$. Let $M = M' \cup H^* \cup V^*$ (if G is a collection of sets, G^* is the union of the sets in G). Fig. 1 gives a sketch which includes some of the points of M. If (a, b) is a component of $I - A$, then there are a total of $4[2b/(b-a)-1]$ segments in $H \cup V$ which lie in the open set $\{(x, y) | a < \max(|x|, |y|) < b\}$, and with the addition of these segments to M', this open set is broken up into $4[2b/(b-a)-1]$ square shaped open sets. Thus each component of $D - M$ is the interior of a square. Another property of M which will be of importance will be pointed out now. Suppose (a, b) is a component of $I - A$ and t is an element of I bigger than b such that no component of $I - A$ longer than (a, b) intersects $[b, t]$, and n is an integer, $|n| < b/(b-a)$. By definition, the interval with endpoints $\big(a, n(b-a)\big)$ and $\big(b, n(b-a)\big)$ lies in M, but it is also true that the interval with endpoints $\big(a, n(b-a)\big)$ and $\big(t, n(b-a)\big)$ lies in M. For suppose $b \leqslant z \leqslant t$. If z is in A, then $\big(z, n(b-a)\big)$ is actually a point of M' since $|n|(b-a) < z$. If z is not in A, then z is in a component (c, d) of $I - A$ which lies to the right of (a, b) and is no longer than (a, b). There will be a non-negative integer k such that $(b-a) = 3^k(d-c)$. $n(b-a) = n3^k(d-c)$, and since $|n| < b/(b-a) = b/3^k(d-c) < d/3^k(d-c)$, it follows that $|n3^k| < d/(d-c)$, so that the interval with endpoints $\big(c, n3^k(d-c)\big)$

and $(d, n3^k(d-c))$ lies in M. Thus z is in M. Similarly, it can be shown that if $t' < a$, and no component of $I-A$ longer than (a, b) intersects $[t', a]$, and $|n|(b-a) < t'$, then the entire interval with endpoints $(t', n(b-a))$ and $(t, n(b-a))$ lies in M. Clearly, it follows that the interval with endpoints $(-t', n(b-a))$ and $(-t, n(b-a))$, the interval with endpoints $(n(b-a), t')$ and $(n(b-a), t)$, and the interval with endpoints $(n(b-a), -t')$ and $(n(b-a), -t)$ also lie in M.

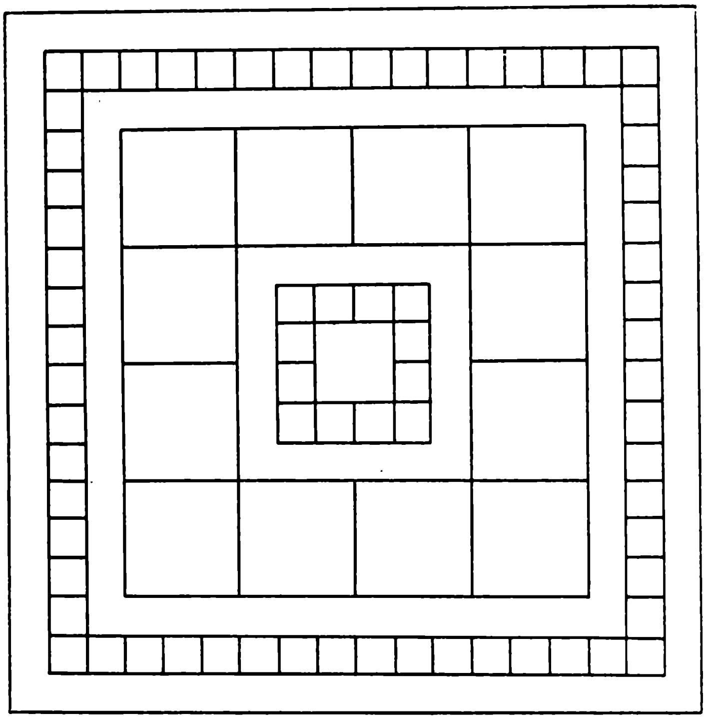

Fig. 1

Now, a monotonic increasing sequence $M_1, M_2, \ldots$ of closed and connected subsets of I^2 will be defined. Let $M_1 = \{(x, y)|$ for some point (u, v) of M, $(x, y) = \frac{1}{2}(u, v)+(\frac{1}{2}, \frac{1}{2})\}$. M_1 has the same configuration with respect to I^2 that M does with respect to D. If n is an integer greater than one, let M_n be the set to which (x, y) belongs if and only if (1) (x, y) is in M_{n-1}, or (2) for some square shaped component S of $I^2 - M_{n-1}$ with center (a, b) and side of length $2d$ and for some point (u, v) of M, $(x, y) = d(u, v)+(a, b)$. Now a certain property of $M_1 \cup M_2 \cup \ldots$ will be established.

Lemma. *Suppose (x, y) is in $M_1 \cup M_2 \cup \ldots$ and interior to I^2, n is the least positive integer such that (x, y) is in M_n, and R is an open set containing (x, y). Then there is a simple closed curve C such that C lies in $M_{n+1} \cap R$ and the interior of C lies in R and contains (x, y).*

Proof. If $Z_1, Z_2, \ldots, Z_n$ is a finite sequence of points, let $C(Z_1, Z_2, \ldots, Z_n)$ denote the set of all points Z which lie on an interval with endpoints Z_i

and Z_{i+1} for some positive integer $i < n$. Let c be a positive number such that $\{(s, t) | \max(|s-x|, |t-y|) < c\}$ lies in R. Assume $n > 1$, and let E be the square component of $I^2 - M_{n-1}$ which contains (x, y) (if $n = 1$, $E = I^2$). Let (a, b) denote the center of E, d the distance from (a, b) to the right side of E, and (u, v) the point of M such that $(x, y) = d(u, v) + (a, b)$. It will be assumed without loss of generality that $0 \leqslant v \leqslant u$. Several cases must be considered.

(1) Suppose u is an element of A but not the end of any component of $I - A$. Let (e, f) be a component of $I - A$ to the left of u such that $u - e < c$ (if $v < u$, also make $u - e < (u - v)/4$) and such that no component of $I - A$ longer than (e, f) intersects $[f, u]$. Then let w be an element of A to the right of u such that $w - u < c$ and no component of $I - A$ longer than (e, f) intersects $[f, w]$. If $u = v$, let Q_1, Q_2, Q_3, and Q_4 be the points (e, e), (w, e), (w, w) and (e, w), respectively. If $u > v$, let n be the largest integer such that $(n-1)(f-e) < v$, and let Q_1, Q_2, Q_3, and Q_4 be $\big(e, (n-1)(f-e)\big)$, $\big(w, (n-1)(f-e)\big)$, $\big(w, (n+1)(f-e)\big)$, and $\big(e, (n+1)(f-e)\big)$, respectively. n will be such that $|n| + 1 < f/(f-e)$. Then let P_1, P_2, P_3, and P_4 be $dQ_1 + (a, b)$, $dQ_2 + (a, b)$, $dQ_3 + (a, b)$, and $dQ_4 + (a, b)$, respectively. The simple closed curve $C(P_1, P_2, P_3, P_4, P_1)$ has the desired properties.

(2) Suppose u is not an element of A. Then the point (u, v) is on one of the segments in H, and there must be two components F and F' of $I^2 - M_n$ such that (x, y) is on their common horizontal edge. Assume F is above F', and let (p, q) be the center of F, r be the distance from (p, q) to the right side of F, and $(w, -1)$ the point of M such that $(x, y) = r(w, -1) + (p, q)$. Let (e, f) be a component of $I - A$ such that $1 - e < \min(c/2, |1-w|/3, |w+1|/3)$ and such that no component of $I - A$ longer than (e, f) intersects $[f, 1]$. Let n be the largest integer such that $(n-1)(f-e) < w$, and let Q_1, Q_2, Q_3, and Q_4 be $\big((n-1)(f-e), -1\big)$, $\big((n-1)(f-e), -f\big)$, $\big((n+1)(f-e), -f\big)$, and $\big((n+1)(f-e), -1\big)$, respectively. Now, let P_1, P_2, P_3, and P_4 be $rQ_1 + (p, q)$, $rQ_2 + (p, q)$, $rQ_3 + (p, q)$, and $rQ_4 + (p, q)$, respectively. P_1 is to the left of (x, y), P_2 is above P_1, P_3 is to the right of P_2 and above P_4, and P_4 is to the right of (x, y). $C(P_1, P_2, P_3, P_4)$ lies in $\mathrm{Cl}(F) \cap M_{n+1} \cap R$. In a similar fashion, P_5, P_6, P_7, and P_8 can be determined so that P_5 is to the right of (x, y), P_6 is below P_5, P_7 is to the left of P_6 and below P_8, P_8 is to the left of (x, y), and $C(P_5, P_6, P_7, P_8)$ lies in $\mathrm{Cl}(F') \cap M_{n+1} \cap R$. Then, $C(P_1, P_2, \ldots, P_8, P_1)$ is the desired simple closed curve.

(3) Suppose u is the left end of a component of $I - A$, and $v < u$. Using methods similar to those in (2), one can determine points P_1, P_2, P_3 and P_4 such that P_1 is above (x, y), P_2 is to the left of P_1, P_3 is below P_2 and to the left of P_4, P_4 is below (x, y), and $C(P_1, P_2, P_3, P_4)$ lies in $E \cap M_n \cap R$. If (x, y) is on the left edge of only one component F of

$I^2 - M_n$, then the techniques used in (2) can be used to determine points P_5, P_6, P_7, and P_8 of $\mathrm{Cl}(F) \cap M_{n+1}$ such that $C(P_1, P_2, \ldots, P_8, P_1)$ will be the desired simple closed curve. Suppose (x, y) is on the left edge of two components F and F' of $I^2 - M_n$. Assume F is the lower one, and let (p, q) be the center of F, and r be the distance from (p, q) to the right edge of F. Let (e, f) be a component of $I - A$ such that $1 - e < c$, and no component of $I - A$ longer than (e, f) intersects $[f, 1]$. Let Q_5, Q_6, and Q_7 be $(-1, e), (-e, e)$, and $(-e, 1)$, respectively, and let P_5, P_6, and P_7 be $rQ_5 + (p, q), rQ_6 + (p, q)$, and $rQ_7 + (p, q)$, respectively. Similarly, one can pick points P_8, P_9, and P_{10} from $\mathrm{Cl}(F')$ so that $C(P_1, P_2, \ldots, P_{10}, P_1)$ is the desired simple closed curve.

(4) If u is the left end of a component of $I - A$ and $u = v$, then the techniques used in (3) can be used four times to determine a sequence $P_1, P_2, \ldots, P_{12}$ such that $C(P_1, P_2, \ldots, P_{12}, P_1)$ will be the desired simple closed curve.

(5) Suppose u is the right end of a component of $I - A$. If $v < u$, the case is almost the same as (3), so suppose $u = v$. Let (e, f) be a component of $I - A$ to the right of u such that $f - u < c$ and no component of $I - A$ longer than (e, f) intersects $[u, e]$. Let n be the largest integer such that $n(f - e) < v$, and let Q_1, Q_2, Q_3, Q_4, and Q_5 be $(u, n(f-e))$, $(f, n(f-e))$, (f, f), $(n(f-e), f)$, and $(n(f-e), u)$, respectively. Let P_1, P_2, P_3, P_4, and P_5 be $dQ_1 + (a, b), dQ_2 + (a, b), dQ_3 + (a, b), dQ_4 + (a, b)$, and $dQ_5 + (a, b)$, respectively. Let F be the component of $I^2 - M_n$ of which (x, y) is the upper right corner. Using the techniques of (3), one can determine points P_6, P_7, and P_8 of $\mathrm{Cl}(F) \cap M_{n+1} \cap R$ such that $C(P_1, P_2, \ldots, P_8, P_1)$ will be the desired simple closed curve.

This completes the proof of the lemma.

Now, for each number t in I, let g_t be a continuous function from I *onto* I which is constant over each component of $I - A$ and such that $g_t(1) = t$. Let $f_1, f_2, \ldots$ be the sequence of functions defined as follows: f_1 has domain M_1 and is such that if (x, y) is a point of M_1 and (u, v) is the point of M such that $(x, y) = \frac{1}{2}(u, v) + (\frac{1}{2}, \frac{1}{2})$, then $f_1(x, y) = g_1[\max(|u|, |v|)]$. Notice that if S is a component of $I^2 - M_1$, then f_1 is constant on $B(S)$. If n is an integer greater than 1, f_n has domain M_n, agrees with f_{n-1} on M_{n-1}, and if (x, y) is a point of $M_n - M_{n-1}$, S is the square component of $I^2 - M_{n-1}$ containing (x, y), (a, b) is the center of S, $2d$ is the length of a side of S, (u, v) is the point of M such that $(x, y) = d(u, v) + (a, b)$, and t is the number which f_{n-1} assigns to every point of $B(S)$, then $f_n(x, y) = g_t[\max(|u|, |v|)]$. The desired function f is defined as follows: if (x, y) is in M_n for some positive integer n, then $f(x, y) = f_n(x, y)$; if (x, y) is in $I^2 - (M_1 \cup M_2 \cup \ldots)$ and $S_1, S_2, \ldots$ is the sequence such that for each positive integer n, S_n is the square component of $I^2 - M_n$ which contains (x, y), then $f(x, y) = \limsup f_n[B(S_n)]$.

Now it will be shown that f is peripherally continuous. Suppose (x, y) is a point of I^2, U is an open set containing (x, y), and V is a segment containing $f(x, y)$, Let $c_1 > 0$ be such that $\{(s, t)|\max(|x-s|, |y-t|) < c_1\}$ lies in U. If (x, y) is in $I^2-(M_1 \cup M_2 \cup \ldots)$ and $S_1, S_2, \ldots$ is the sequence such that for each positive integer n, S_n is the component of I^2-M_n which contains (x, y), then there is a positive integer n such that S_n is a subset of U and $f[B(S_n)]$ is in V. S_n will be the desired open set W. On the other hand, suppose n is the least positive integer such that (x, y) is in M_n. In case $n > 1$, let E be the square component of I^2-M_{n-1} which contains (x, y) (if $n = 1$, let $E = I^2$). f_n is continuous on $\mathrm{Cl}(E) \cap M_n$, so let $c_2 > 0$ be such that if (s, t) is in $\mathrm{Cl}(E) \cap M_n$ and $\max(|x-s|, |y-t|) < c_2$, then $f_n(s, t)$ is in V. First, suppose (x, y) is on an edge of I^2. Let $R' = \{(s, t)|\max(|x-s|, |y-t|) < \min(c_1, c_2)\}$. If (x, y) is not a corner of I^2, then the technique of part (2) of the proof of the lemma will determine four points P_1, P_2, P_3, and P_4 with (x, y) between P_1 and P_4 on an edge of I^2 such that $C(P_1, P_2, P_3, P_4)$ lies in $R' \cap M_1$ and is the boundary with respect to I^2 of an open (with respect to I^2) set W which contains (x, y). W will be the desired open set. If (x, y) is a corner of I^2, then the techniques of part (3) of the proof of the lemma will determine three points P_1, P_2, and P_3 such that $C(P_1, P_2, P_3)$ will be the boundary with respect to I^2 of the desired open set W. Now, suppose (x, y) is not on an edge of I^2. If (x, y) is also on the edge of some square component of I^2-M_n, then let $c_3 > 0$ be such that if F is a component of I^2-M_n with (x, y) on its edge and (s, t) is a point of $\mathrm{Cl}(F) \cap M_{n+1}$ such that $\max(|x-s|, |y-t|) < c_3$, then $f_{n+1}(s, t)$ is in V. Let $c = \min(c_1, c_2, c_3)$ and let $R = \{(s, t)| \max(|x-s|, |y-t|) < c\}$. Now, let C be the appropriate simple closed curve constructed in the proof of the lemma. The interior W of C contains (x, y), lies in U, and $f[B(W)]$ is a subset of V. Thus f is peripherally continuous and a connectivity function.

Since the real functions g_t used in the construction of f were from I *onto* I, the function f is clearly dense in $I^2 \times I$.

Comments. An argument similar to that given for the theorem in [1] will show that if f is a connectivity function from I^2 into I and f is totally discontinuous, then its graph must be dense in some open subset of $I^2 \times I$. Therefore, in constructing a totally discontinuous connectivity function f from I^2 into I, any difficulty incurred in actually making f dense in $I^2 \times I$ is unavoidable.

In [3] Cornette showed that there is a space (an explosion set in the plane) which is the range of a connectivity function with domain I but not the range of a connectivity function with domain I^2, and in a paper given at the 1968 conference on point set topology at The University of Houston he raised the question as to whether it is true that if n is a positive integer, there is a space which is the range of a connectivity function with domain I^n but not the range of a connectivity function with domain I^{n+1}. That question is not answered in this paper, but consider the following

THEOREM. *If f is a connectivity function from I into I which is dense in $I \times I$, then the graph of f is the range of a connectivity function with domain I but not the range of a connectivity function with domain I^2.*

Proof. Suppose G is the graph of a connectivity function from I into I which is dense in $I \times I$ and f is a connectivity function from I^2 onto G, Suppose x is a point of I^2 and R_1 and R_2 are circular regions of radius $\frac{1}{4}$ with centers x and $f(x)$, respectively. Notice that $R_2 \cap G$ is totally disconnected. Since f is peripherally continuous, there is an open subset R of R_1 which contains (x, y) and is such that $f[B(R)]$ is a subset of R_2. $B(R)$ must have a non-degenerate component C'. $f(C')$ is a connected subset of $R_2 \cap G$, so it must contain just one point z. Let C be the component of $f^{-1}(z)$ which has C' as a subset. From Theorem 2 of [4] it follows that C is closed. Let y be a point of $I^2 - C$, d_1 be the minimum distance from y to a point of C, w be a point of C at a distance d_1 from y, and d_2 be the diameter of C. Let U be a circular region with center w and radius less than $\frac{1}{2}d_1$ and $\frac{1}{2}d_2$. Let V be an open subset of U which contains w and is such that $f[B(V)]$ is a subset of R_2. There is a connected open set V' which contains w, lies in U and has a connected boundary which is a subset of $B(V)$. Since $B(V')$ is connected and $f[B(V')]$ lies in $R_2 \cap G$, $f[B(V')]$ must contain just one point; and since $B(V')$ intersects C, $f[B(V')] = (z)$. Therefore, $C \cup B(V')$ is a connected subset of $f^{-1}(z)$, but it contains a point on the segment from w to y which does not belong to C. This is a contradiction.

Since G is a connected separable metric space, it follows from Cornette's theorem that it is the range of a connectivity function with domain I.

Since it has been established that there is a connectivity function from I^2 into I which is dense in $I^2 \times I$, it would be interesting if the previous theorem could be extended to show that the graph of such a function is not the range of a connectivity function with domain I^3 (it is the range of a connectivity function with domain I^2). This author has not even been able to show that the particular example constructed in this paper has this property. (**P 716**)

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