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ON Γ-REGULAR GRAPHS

BY

J. PŁONKA (WROCŁAW)

0. We accept the terminology and definitions from the book by Harary ([1]). In particular, by a *graph* we mean a pair $G = (V; X)$, where V is a non-empty set called the *set of vertices* and X is a set of 2-element subsets of V called the *set of edges*. We do not require V to be finite. Two vertices v_1 and v_2 are called *adjacent* if $\{v_1, v_2\} \in X$; in that case we write $v_1 \leftrightarrow v_2$. By a *subgraph* of G we mean any of the graphs $(V_0; X_0)$, where $\emptyset \neq V_0 \subseteq V$ and $X_0 = \{\{v_1, v_2\} \in X\colon v_1, v_2 \in V_0\}$. A sequence $v_1, \ldots, v_n$ of different elements of V is called a *simple chain* from v_1 to v_n if $n = 1$ or $n > 1$ and $\{v_i, v_{i+1}\} \in X$ for $1 \leqslant i < n$. A graph G is called *connected* if for any two vertices $v, v' \in V$ there exists a simple chain from v to v'. A maximal connected subgraph of G is called a *component* of G. The graph G is called *bipartite* if $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$, $V_1 \neq \emptyset \neq V_2$ and $\{v_1, v_2\} \in X \Rightarrow v_1 \in V_1, v_2 \in V_2$, or $v_1 \in V_2, v_2 \in V_1$. For $v \in V$ we write

$$\Gamma(v) = \{u\colon u \leftrightarrow v,\ u \in V\}.$$

The number $\varrho(v) = |\Gamma(v)|$ will be called the *degree* of v. A graph $G = (V; X)$ is called *k-regular* ($k \geqslant 0$) if $\varrho(v) = k$ for each $v \in V$.

In this paper we study a more general notion of regularity ([2]). Namely, for $v \in V$ we define

$$\varrho_\Gamma(v) = \begin{cases} \sum_{u \in \Gamma(v)} \varrho(u) & \text{if } \Gamma(v) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a graph $G = (V; X)$ is *m-Γ-regular* ($m \geqslant 0$ and m is an integer) if $\varrho_\Gamma(v) = m$ for each $v \in V$. Let us say that a bipartite graph

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([1]) F. Harary, *Graph theory*, Addison-Wesley, 1969.

([2]) The same notion has been independently introduced also in S. Rama Chadran, *Nearly regular graphs and their reconstruction*, Graph Theory Newsletter 8 (1978), p. 3 (Note of the Editors).

$(V_1 \cup V_2; X)$ is *(k, s)-regular* if for each $v \in V_1$ we have $\varrho(v) = k$ and for each $u \in V_2$ we have $\varrho(u) = s$.

In this paper we give a characterization of m-Γ-regular graphs.

For a real number r we denote by $[r]$ the integer part of r.

1. Observe first that if $G = (V; X)$ is a finite graph, then

(i) $\sum_{v \in V} \varrho_\Gamma(v) = \sum_{v \in V} \varrho^2(v)$.

Indeed, since $v \in \Gamma(u)$ for any $u \in \Gamma(v)$, counting the left-hand side of (i) we take the number $\varrho(v)$ as many times as many elements the set $\Gamma(v)$ contains, i.e. $\varrho(v)$ times. From (i) we get

(ii) if a graph $G = (\{v_1, \ldots, v_n\}; X)$ is m-Γ-regular, then

$$m = \frac{\varrho^2(v_1) + \ldots + \varrho^2(v_n)}{n}.$$

A vertex v of a graph $G = (V; X)$ is called *Γ-regular* if for each $u, w \in \Gamma(v)$ we have $\varrho(u) = \varrho(w)$. If v is not Γ-regular, we say that v is *non-Γ-regular*.

LEMMA 1. *If a graph $G = (V; X)$ is m-Γ-regular for some $m > 0$ and there exists a non-Γ-regular vertex $v \in V$ such that $\varrho(v) = k$, then there exists a non-Γ-regular vertex $v' \in V$ such that $\varrho(v') = k' < k$.*

Proof. Since v is non-Γ-regular, we have $k > 0$. Moreover, it follows that $1 < k < m$. In fact, if $k = 1$, then v is regular; if $k = m$, then for any $w \in \Gamma(v)$ we have $\varrho(w) = 1$ and v is regular again. Put $s = [m/k]$. Then

$$ks \leqslant m, \tag{1}$$

and if q is a positive integer, then

$$k(s+q) > m. \tag{2}$$

Let $\Gamma(v) = \{v_1, \ldots, v_{\varrho(v)}\}$. Since G is m-Γ-regular, we have

$$\varrho(v_1) + \ldots + \varrho(v_{\varrho(v)}) = m. \tag{3}$$

But v is non-Γ-regular, so there exists $v_i \in \Gamma(v)$ such that $\varrho(v_i) = s+q$ for some positive integer q. In fact, if $\varrho(v_j) = s$ for $j = 1, \ldots, \varrho(v)$, then v is regular — a contradiction. If $\varrho(v_j) \leqslant s$ for $j = 1, \ldots, \varrho(v)$ and $\varrho(v_t) < s$ for some $t \in \{1, \ldots, \varrho(v)\}$, then by (1) we get a contradiction with (3). Put

$$k' = \min\{\varrho(w): w \in \Gamma(v_i)\}.$$

Let $v' \in \Gamma(v_i)$ be a vertex of V such that $\varrho(v') = k'$. Since $v \in \Gamma(v_i)$, $\varrho(v_i) = s+q$, $\varrho(v) = k$, so by (2) and the m-Γ-regularity of G we obtain $k' < k$. We have also

$$k' \leqslant \frac{m-k}{s+q-1}. \tag{4}$$

In fact, let $\Gamma(v_i) = \{v, w_1, \ldots, w_{s+q-1}\}$. Then

$$\varrho(v)+\varrho(w_1)+\ldots+\varrho(w_{s+q-1}) = m$$

and

$$k' \leqslant \min\{\varrho(v), \varrho(w_1), \ldots, \varrho(w_{s+q-1})\} \leqslant \min\{\varrho(w_1), \ldots, \varrho(w_{s+q-1})\}$$
$$\leqslant \frac{\varrho(w_1)+\ldots+\varrho(w_{s+q-1})}{s+q-1} = \frac{m-k}{s+q-1}.$$

We have further

$$\frac{m-k}{s+q-1}(s+q) < m. \tag{5}$$

In fact, assume

$$\frac{m-k}{s+q-1}(s+q) \geqslant m.$$

Then $-k(s+q) \geqslant -m$ and $k(s+q) \leqslant m$, which contradicts (2). By (4) and (5) we have

$$k'(s+q) \leqslant \frac{m-k}{s+q-1}(s+q) < m.$$

Thus

$$k'(s+q) < m. \tag{6}$$

Now we can prove that v' is non-Γ-regular. Assume that v' is Γ-regular. We have $\varrho(v') = k'$ and $v_i \in \Gamma(v')$. Consequently, since G is m-Γ-regular, we get $\varrho_\Gamma(v') = k'(s+q) = m$, which contradicts (6).

COROLLARY. *If a graph $G = (V; X)$ is m-Γ-regular for $m \geqslant 0$, then any vertex $v \in V$ is Γ-regular.*

Proof. For $m = 0$ the proof is obvious. If $m > 0$, then by Lemma 1 all vertices of G have to be Γ-regular. Otherwise, using Lemma 1 we obtain an infinite sequence $v, v', v'', \ldots$ of non-Γ-regular vertices such that $\varrho(v) > \varrho(v') > \varrho(v'') > \ldots$, which is impossible. Let us recall that if $\varrho(u) = 1$ or $\varrho(u) = 0$, then u is Γ-regular.

LEMMA 2. *If in a connected graph $G = (V; X)$ any vertex is Γ-regular and for some vertex v we have $\varrho_\Gamma(v) = m \geqslant 0$, where m is an integer, then*

G is either k-regular, where $k = \sqrt{m}$, or G is a bipartite (k, s)-regular graph, where $ks = m$.

Proof. If $m = 0$, then $V = \{v\}$, $X = \emptyset$, and G is 0-regular. Assume $m > 0$. Put $\varrho(v) = k$. So for $w \in \Gamma(v)$ we have $\varrho(w) = m/k = s$. Let v' be a vertex of G different from v. Since G is connected, there exists a simple chain from v to v'. Let $v = v_1, v_2, \ldots, v_p = v'$ be such a chain. Since v_2 is Γ-regular and $v_1, v_3 \in \Gamma(v_2)$, we obtain $\varrho(v_3) = \varrho(v_1) = \varrho(v) = k$. Since v_3 is Γ-regular, we have $\varrho(v_2) = \varrho(v_4) = s$. In general, $\varrho(v_{2r}) = s$, $\varrho(v_{2r-1}) = k$ $(r = 1, \ldots, [p+1])$. If $k = s$, then $\varrho(v') = k$ and G is k-regular, where $k^2 = m$. Let $k \neq s$. If p is odd, then $\varrho(v') = k$, and since $v' \leftrightarrow v_{p-1}$ and $\varrho(v_{p-1}) = s$, so by Γ-regularity of v' we have $\varrho(w) = s$ for each $w \in \Gamma(v')$. Thus v' is adjacent only to k vertices having degrees equal to s. Analogously, if p is even, then v' is adjacent only to s vertices having degrees equal to k. Now it is enough to put $V_1 = \{u : \varrho(u) = k\}$ and $V_2 = \{w : \varrho(w) = s\}$ to see that G is a bipartite (k, s)-regular graph and $ks = m$. Thus the proof is complete.

Let m be a non-negative integer.

THEOREM. *A graph G is m-Γ-regular if and only if each of the components of G is either a k-regular subgraph of G, where $k^2 = m$, or a (k, s)-regular bipartite subgraph of G, where $ks = m$.*

Proof. The proof of the sufficiency is obvious. The necessity follows from the Corollary and Lemma 2.

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