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ON SECTORIAL QUALITATIVE CLUSTER SETS AND DIRECTIONAL QUALITATIVE CLUSTER SETS

BY

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1. Let H denote the open upper half plane above the real line R and let z and x denote points in H and R respectively. We write $\bar{A}$ to denote the closure of the set A. For two directions θ and φ, $0 < \theta < \varphi < \pi$, we define

$$S_{\theta\varphi} = \{z\colon z \in H;\ \theta < \arg(z) < \varphi\}.$$

Then $S_{\theta\varphi}$ is a sector in H with vertex at the origin. By $S_{\theta\varphi}(x)$ we denote the translate of $S_{\theta\varphi}$ which is obtained by taking the origin at x. If there is no confusion then we simply write S and $S(x)$ instead of $S_{\theta\varphi}$ and $S_{\theta\varphi}(x)$. For fixed $x \in R$, fixed $\theta \in (0, \pi)$ and $r > 0$ we write

$$L_\theta(x) = \{z\colon z \in H;\ \arg(z-x) = \theta\},$$

$$K(x, r) = \{z\colon z \in H;\ |z-x| < r\},$$

$$S(x, r) = S(x) \cap K(x, r)$$

and

$$L_\theta(x, r) = L_\theta(x) \cap K(x, r).$$

Let $f\colon H \to W$, where W is a topological space. Then the *qualitative cluster set* $C_q(f, x)$ of f at x is the set of all $w \in W$ such that for every open set U in W containing w, the set $f^{-1}(U) \cap K(x, r)$ is of the second category for each $r > 0$. The *sectorial qualitative cluster set* $C_q(f, x, S)$ of f at x in the sector S is the set of all $w \in W$ such that for every open set U in W containing w, $f^{-1}(U) \cap S(x, r)$ is of the second category for all $r > 0$. The *directional qualitative cluster set* $C_q(f, x, \theta)$ of f at x in the direction θ is the set of all $w \in W$ such that for every open set U in W containing w, $f^{-1}(U) \cap L_\theta(x, r)$ is of the second category in $L_\theta(x)$ for every $r > 0$.

Following the definition of Dolzhenko [1], a set $E \subset R$ is said to be *porous at a point* $x \in R$ if

$$\limsup_{\gamma \to 0} \frac{\gamma(x, r, E)}{r} > 0,$$

where $\gamma(x, r, E)$ is the length of the largest open interval in the complement of E which is entirely contained in $(x-r, x+r)$. A set $E \subset R$ is said to be *porous* if it is porous at each of its points. A set $E \subset R$ is said to be *σ-porous* if it is a countable union of porous sets. Then clearly a subset of a σ-porous set is a σ-porous set and a σ-porous set is a first category set of measure zero. But Zajíček [5] has constructed a perfect set of measure zero which is not a σ-porous set.

A set $E \subset H$ is said to *have Baire property* if there are an open set G and a first category set Q in H such that

$$E = G \Delta Q = (G \setminus Q) \cup (Q \setminus G).$$

A function $f\colon H \to W$, where W is a topological space, is said to *have Baire property* if for every open set G in W the set $f^{-1}(G)$ has Baire property.

2. Wilczyński [4] proved that if $f\colon H \to R$ has Baire property and if θ, $0 < \theta < \pi$, is a fixed direction then, except a first category set of points x in R,

$$C_q(f, x) = C_q(f, x, \theta).$$

Evans and Humke [2] have proved an analogue of this result which states that if $f\colon H \to R$ has Baire property then, except a first category set of points x in R, the set

$$\{\theta\colon\ 0 < \theta < \pi,\ C_q(f, x) = C_q(f, x, \theta)\}$$

is residual in $(0, \pi)$.

In connection with above results we prove the following results:

(i) *If $f\colon H \to W$ has Baire property, where W is a second countable topological space, and if $\{S\}$ is the collection of all sectors in H then, except at a σ-porous set of points x in R, the set*

$$\{\theta\colon\ 0 < \theta < \pi;\ \bigcup\{C_q(f, x, S)\colon\ S \in \{S\}\} \subset C_q(f, x, \theta)\}$$

is residual in $(0, \pi)$.

(ii) *If $f\colon H \to W$ is arbitrary, where W is a second countable topological space, and if $\{S\}$ is the collection of all sectors in H then at each point x in R the set*

$$\{\theta\colon\ 0 < \theta < \pi;\ C_q(f, x, \theta) \subset \bigcup\{C_q(f, x, S)\colon\ S \in \{S\}\}\}$$

is residual in $(0, \pi)$.

3. Prior to the proof of the auxiliary lemmas we shall define some sets and establish relations among them which will be used in the sequel. For fixed $x \in R$ and a set $P \subset H$ let

$$O(P, x) = \{\theta\colon\ 0 < \theta < \pi;\ P \cap L_\theta(x, r) \text{ is of the first category in } L_\theta(x) \text{ for at least one } r > 0\},$$

$$T(P, x) = \{S\colon\ S \subset H;\ S(x, r) \cap P \text{ is of the second category for all } r > 0\}$$

and for rationals α, β, $0 < \alpha < \beta < \pi$, let

$$T_{\alpha\beta}(P, x) = \{S\colon\ \bar{S} \subset H \setminus \bar{S}_{\alpha\beta};\ S(x, r) \cap P \text{ is of the second category for all } r > 0\}.$$

Let $S_{\theta\varphi} \subset H$ be a fixed sector. Then we write for $P \subset H$ and $x \in R$,

$$\hat{T}_{\theta\varphi}(P, x) = \{S\colon\ S \subset S_{\theta\varphi};\ S(x, r) \cap P \text{ is of the second category for all } r > 0\}.$$

Furthermore, for a fixed positive integer n we define for any set $P \subset H$ and fixed $x \in R$,

$$O_n(P, x) = \{\theta\colon\ 0 < \theta < \pi;\ L_\theta(x, 1/n) \cap P \text{ is of the first category in } L_\theta(x)\}.$$

Then, clearly,

$$O(P, x) \subset \bigcup_{n=1}^{\infty} O_n(P, x). \tag{1}$$

LEMMA 1. *Let $G \subset H$ be open. Then the set*

$$B(G) = \{x\colon\ x \in R;\ T(G, x) \neq \emptyset,\ O(G, x) \text{ is of the second category in } (0, \pi)\}$$

is a σ-porous set.

Proof. For a positive integer n and rationals α, β, $0 < \alpha < \beta < \pi$, let

$$B_{n\alpha\beta}(G) = \{x\colon\ x \in R;\ T_{\alpha\beta}(G, X) \neq \emptyset,\quad O_n(G, x) \text{ is dense in } (\alpha, \beta)\}.$$

Now, if $x_0 \in B(G)$ then $T(G, x_0) \neq \emptyset$ and $O(G, x_0)$ is of the second category, so the relation (1) ensures that there is n_0 such that $O_{n_0}(G, x_0)$ is dense in some open subset of $(0, \pi)$. Thus we can find two rationals γ, δ, $0 < \gamma < \delta < \pi$, such that $O_{n_0}(G, x_0)$ is dense in (γ, δ). Again, since G is open therefore, for $\theta \in O_{n_0}(G, x_0)$, $L_\theta(x_0, 1/n_0) \cap G = \emptyset$. Thus G is open and $O_{n_0}(G, x_0)$ is dense in (γ, δ) imply that for all $\theta \in [\gamma, \delta]$, $L_\theta(x_0, 1/n_0) \cap G = \emptyset$. Let

$$S_{\gamma\delta} = \{z\colon\ z \in H;\ \gamma < \arg(z) < \delta\}.$$

Then, clearly, from the above argument

$$S_{\gamma\delta}(x_0, 1/n_0) \cap G = \emptyset.$$

Again $T(G, x_0) \neq \emptyset$ implies that there is S^0 such that $S^0(x_0, r) \cap G$ is of the second category for all $r > 0$. Thus there exists a sector $S' \subset S^0 \setminus \bar{S}_{\gamma\delta}$ such that $S'(x_0, r) \cap G$ is of the second category for all $r > 0$. Let α_0, β_0 be two rationals such that $\gamma < \alpha_0 < \beta_0 < \delta$. Then, clearly, $\bar{S}' \subset H \setminus \bar{S}_{\alpha_0\beta_0}$, i.e. $T_{\alpha_0\beta_0}(G, x_0) \neq \emptyset$ and $O_{n_0}(G, x_0)$ is dense in (α_0, β_0). Hence $x_0 \in B_{n_0\alpha_0\beta_0}(G)$ and, consequently,

$$B(G) \subset \bigcup_n \bigcup_\alpha \bigcup_\beta B_{n\alpha\beta}(G). \tag{2}$$

Suppose for some positive integer n and rationals α, β, $0 < \alpha < \beta < \pi$, the set $B_{n\alpha\beta}(G)$ is non-porous. Then there is $x_0 \in B_{n\alpha\beta}(G)$ such that

$$\lim_{r \to 0} \frac{\gamma(x_0, r, B_{n\alpha\beta}(G))}{r} = 0, \tag{3}$$

where $\gamma(x_0, r, B_{n\alpha\beta}(G))$ is the length of the greatest interval in the complement $B_{n\alpha\beta}(G)$ and which is entirely contained in $(x_0 - r, x_0 + r)$. As $x_0 \in B_{n\alpha\beta}(G)$, $T_{\alpha\beta}(G, x_0) \neq \emptyset$. Let $S^0 \in T_{\alpha\beta}(G, x_0)$. If $S^0 = \{z\colon z \in H;\ \gamma < \arg(z) < \delta\}$ then either $0 < \alpha < \beta < \gamma < \delta < \pi$ or $0 < \gamma < \delta < \alpha < \beta < \pi$. Let us suppose that $0 < \alpha < \beta < \gamma < \delta < \pi$. Let

$$K = \frac{\sin\delta \sin(\beta - \alpha)}{\sin\beta \sin(\delta - \alpha)}.$$

Then from (3) we infer that, for arbitrary ε, $0 < \varepsilon < K/2$, there is $\eta > 0$ such that

$$\gamma(x_0, r, B_{n\alpha\beta}(G)) < \varepsilon r \quad \text{for all } r < \eta. \tag{4}$$

Let $x' \in B_{n\alpha\beta}(G)$. Then, since G is open and $O_n(G, x')$ is dense in (α, β), therefore $[\alpha, \beta] \subset O_n(G, x')$, i.e. for every $\theta \in [\alpha, \beta]$, $L_\theta(x', 1/n) \cap G = \emptyset$. Thus $S_{\alpha\beta}(x', 1/n) \cap G = \emptyset$, where $S_{\alpha\beta} = \{z\colon z \in H;\ \alpha < \arg(z) < \beta\}$. Since x' is an arbitrary point of $B_{n\alpha\beta}(G)$, we have

$$S_{\alpha\beta}(x, 1/n) \cap G = \emptyset \quad \text{for all } x \in B_{n\alpha\beta}(G). \tag{5}$$

Since $B_{n\alpha\beta}(G)$ is non-porous at x_0, for all $x < x_0$, $(x, x_0) \cap B_{n\alpha\beta}(G) \neq \emptyset$. Let $\xi \in (x_0 - \eta, x_0) \cap B_{n\alpha\beta}(G)$ be such that $S_{\alpha\beta}(\xi, 1/n) \cap S^0(x_0)$ is a quadrilateral. Since $S^0(x_0, r) \cap G \neq \emptyset$ for all $r > 0$, therefore we can choose a ζ such that $\xi < \zeta < x_0$ and $L_\alpha(\zeta, 1/n)$ intersects $S^0(x_0) \cap G$. Let $z_0 \in L_\alpha(\zeta, 1/n) \cap S^0(x_0) \cap G$ and let

$$S_{\alpha\beta}(\zeta, 1/n) \cap L_\delta(x_0) = J(\zeta).$$

Then $J(\zeta)$ is an open segment on $L_\delta(x_0)$ with end points at $L_\alpha(\zeta, 1/n) \cap L_\delta(x_0)$ and $L_\beta(\zeta, 1/n) \cap L_\delta(x_0)$ and let $I(\zeta)$ be the open interval on R with left end point at ζ and

$$|I(\zeta)| = |J(\zeta)|\frac{\sin(\delta-\beta)}{\sin\beta},$$

where $|\cdot|$ denotes the length of the interval. Then, clearly,

$$\frac{|I(\zeta)|}{x_0-\zeta} = K. \tag{6}$$

Since, from (4), $\gamma(x_0, x_0-\zeta, B_{n\alpha\beta}(G)) < (x_0-\zeta)\varepsilon$ and $0 < \varepsilon < K/2$, equality (6) ensures that there is a point $x' \in B_{n\alpha\beta}(G) \cap I(\zeta)$. Clearly $z_0 \in S_{\alpha\beta}(x', 1/n)$.

Since $z_0 \in G$, $S_{\alpha\beta}(x', 1/n) \cap G \neq \emptyset$. But this contradicts (5) since $x' \in B_{n\alpha\beta}(G)$. Considering the case $0 < \gamma < \delta < \alpha < \beta < \pi$ one can arrive at a similar contradiction by proceeding from the right of x_0. Thus each of the sets $B_{n\alpha\beta}(G)$ is porous and hence by (2) $B(G)$ is a σ-porous set.

LEMMA 2. *Let $E \subset H$ have Baire property. Then the set*

$$B(E) = \{x\colon x \in R;\ T(E, x) \neq \emptyset,\ O(E, x) \text{ is of the second category in } (0, \pi)\}$$

is a σ-porous set.

Proof. Since E has the property of Baire, therefore there are open set G and a first category set Q in H such that

$$E = G \mathbin{\Delta} Q = (G \setminus Q) \cup (Q \setminus G).$$

Now, since $G \setminus Q \subset E$, we have

$$O(E, x) \setminus O(G \setminus Q, x) = \emptyset. \tag{7}$$

Again, since

$$O(G, x) = \{\theta\colon 0 < \theta < \pi;\ L_\theta(x, r) \cap G \text{ is of the first category in } L_\theta(x) \text{ for at least one } r > 0\},$$

$$O(G \setminus Q, x) = \{\theta\colon 0 < \theta < \pi;\ L_\theta(x, r) \cap (G \setminus Q) \text{ is of the first category in } L_\theta(x) \text{ for at least one } r > 0\},$$

$$L_\theta(x, r) \cap G = [L_\theta(x, r) \cap (G \setminus Q)] \cup [L_\theta(x, r) \cap (Q \cap G)]$$

and by the Kuratowski–Ulam Theorem ([3], p. 56) the set

$$\{\theta\colon 0 < \theta < \pi;\ L_\theta(x) \cap (Q \cap G) \text{ is of the second category in } L_\theta(x)\}$$

is a first category set, therefore the set $O(G \setminus Q, x) \setminus O(G, x)$ is a first category set. Hence by (7) and by the fact that $O(G \setminus Q, x) \setminus O(G, x)$ is a first category set, it follows that

$$O(E, x) \setminus O(G, x) \text{ is a first category set.} \tag{8}$$

Let $x' \in B(E)$. Then $T(E, x') \neq \emptyset$ and $O(E, x')$ is of the second category in $(0, \pi)$. Since $T(E, x') \neq \emptyset$ and $O(E, x')$ is of the second category in $(0, \pi)$, therefore $T(G, x') \neq \emptyset$ and, by (8), $O(G, x')$ is also of the second category in $(0, \pi)$. Thus $x' \in B(G)$. Hence

$$B(E) \subset B(G). \tag{9}$$

By Lemma 1 $B(G)$ is a σ-porous set and hence, by (9), $B(E)$ is a σ-porous set.

LEMMA 3. *Let $E \subset H$ be arbitrary and let $x \in R$ be fixed. Let $S_{\theta_1\theta_2} \subset H$ be a fixed sector. Then, if the set $\hat{T}_{\theta_1\theta_2}(E, x) = \emptyset$, the set $(\theta_1, \theta_2) \setminus O(E, x)$ is of the first category in* $(0, \pi)$.

Proof. For a fixed positive integer n and rationals α, β, $\theta_1 < \alpha < \beta < \theta_2$, let

$$\Lambda_{n\alpha\beta}(x) = \{\theta\colon\ \alpha < \theta < \beta;\ L_\theta(x, r) \cap E \text{ is of the second category in } L_\theta(x) \text{ for all } r,\ 0 < r < 1/n\}.$$

Then clearly

$$(\theta_1, \theta_2) \setminus O(E, x) \subset \bigcup_n \bigcup_\alpha \bigcup_\beta \Lambda_{n\alpha\beta}(x).$$

If possible, for positive integer n and rationals α, β, $\theta_1 < \alpha < \beta < \theta_2$, let $\Lambda_{n\alpha\beta}(x)$ be of the second category. Then by the Kuratowski–Ulam Theorem ([3], p. 56), $S_{\alpha\beta}(x, r) \cap E$ is of the second category for $0 < r < 1/n$. Hence we have $S_{\alpha\beta} \in \hat{T}_{\theta_1\theta_2}(E, x)$, which contradicts the fact that $\hat{T}_{\theta_1\theta_2}(E, x) = \emptyset$. Hence the set $\Lambda_{n\alpha\beta}(x)$ is of the first category for all positive integers n and rationals α, β, $\theta_1 < \alpha < \beta < \theta_2$. Thus the set $(\theta_1, \theta_2) \setminus O(E, x)$ is of the first category.

THEOREM 1. *If $f\colon H \to W$ has Baire property, where W is a second countable topological space, and if $\{S\}$ is the collection of all sectors in H, then, except a σ-porous set of points x in R, the set*

$$O(x) = \{\theta\colon\ 0 < \theta < \pi;\ \bigcup\{C_q(f, x, S)\colon\ S \in \{S\}\} \subset C_q(f, x, \theta)\}$$

is residual in $(0, \pi)$.

Proof. Let $B = \{V_n\}$ be a countable basis for the topology of W. Also let

$$E_n = f^{-1}(V_n),$$

$$B(E_n) = \{x\colon\ x \in R;\ T(E_n, x) \neq \emptyset,\ O(E_n, x) \text{ is of the second category in } (0, \pi)\}$$

and

$$T = \{x\colon\ x \in R;\ (0, \pi) \setminus O(x) \text{ is of the second category in } (0, \pi)\}.$$

Let $\theta_0 \in (0, \pi)\setminus O(x)$. Then there is $S^0 \in \{S\}$ such that

$$C_q(f, x, S^0) \not\subset C_q(f, x, \theta_0).$$

Thus there is $\omega_0 \in C_q(f, x, S^0)$ such that $\omega_0 \notin C_q(f, x, \theta_0)$. So there is n_0 such that $S^0(x, r) \cap E_{n_0}$ is of the second category for all $r > 0$ and $L_{\theta_0}(x, r) \cap E_{n_0}$ is of the first category in $L_{\theta_0}(x)$ for at least one $r > 0$. Hence $S^0 \in T(E_{n_0}, x)$ and $\theta_0 \in O(E_{n_0}, x)$. Thus we have proved

$$(0, \pi)\setminus O(x) \subset \bigcup O(E_n, x), \tag{10}$$

where the union will be taken for all positive integers n for which $T(E_n, x) \neq \emptyset$. Let $y \in T$. Then $(0, \pi)\setminus O(y)$ is of the second category and hence by (10) there is n_0 such that $O(E_{n_0}, y)$ is of the second category and $T(E_{n_0}, y) \neq \emptyset$.

Hence $y \in B(E_{n_0})$ and consequently,

$$T \subset \bigcup_{n=1}^{\infty} B(E_n). \tag{11}$$

Now, by Lemma 2, it follows that each of the sets $B(E_n)$ is a σ-porous set and so by (11) T is a σ-porous set. This completes the proof of Theorem 1.

THEOREM 2. *If $f\colon H \to W$ is arbitrary, where W is a second countable topological space, and if $\{S\}$ is the collection of all sectors in H then at every x in R the set*

$$\vartheta(x) = \{\theta\colon\ 0 < \theta < \pi;\ C_q(f, x, \theta) \subset \bigcup\{C_q(f, x, S)\colon\ S \in \{S\}\}\}$$

is residual in $(0, \pi)$.

Proof. Let $B = \{V_n\}$ be a countable basis for the topology of W. Let

$$E_n = f^{-1}(V_n)$$

and let for rationals α, β, $0 < \alpha < \beta < \pi$,

$$\Phi_{\alpha\beta}(x) = (\alpha, \beta)\setminus \vartheta(x),$$

$$\hat{T}_{\alpha\beta}(E_n, x) = \{S\colon\ S \subset S_{\alpha\beta};\ S(x, r) \cap E_n \text{ is of the second category for all } r > 0\}.$$

Then

$$(0, \pi)\setminus \vartheta(x) \subset \bigcup_{\alpha} \bigcup_{\beta} \Phi_{\alpha\beta}(x). \tag{12}$$

Now if $\theta_0 \in \Phi_{\alpha\beta}(x)$, then $C_q(f, x, \theta_0) \not\subset C_q(f, x, S_{\alpha\beta})$. Therefore there is $\omega_0 \in C_q(f, x, \theta_0)\setminus C_q(f, x, S_{\alpha\beta})$. Thus there is n_0 such that $\theta_0 \in (\alpha, \beta)\setminus O(E_n, x)$ and $\hat{T}_{\alpha\beta}(E_{n_0}, x) = \emptyset$. These imply that

$$\Phi_{\alpha\beta}(x) \subset \bigcup_{n} [(\alpha, \beta)\setminus O(E_n, x)]$$

where the union is taken for all positive integers n for which $\hat{T}_{\alpha\beta}(E_n, x) = \emptyset$.

Now, by Lemma 3, each of the sets $(\alpha, \beta)\setminus O(E_n, x)$ is of the first category whenever $\hat{T}_{\alpha\beta}(E_n, x) = \emptyset$. Thus $\Phi_{\alpha\beta}(x)$ is of the first category. Hence, by (12), the set $(0, \pi)\setminus \vartheta(x)$ is of the first category, i.e. $\vartheta(x)$ is residual in $(0, \pi)$. This completes the proof of the theorem.

COROLLARY 1. *If f: $H \to W$ has Baire property, where W is a second countable topological space, and if $\{S\}$ is the collection of all sectors in H then, except a σ-porous set of points x in R, the set*

$$\vartheta(x) = \{\theta\colon\ 0 < \theta < \pi;\ C_q(f, x, \theta) = \bigcup\{C_q(f, x, S)\colon\ S \in \{S\}\}\}$$

is residual in $(0, \pi)$.

Proof follows from Theorem 1 and Theorem 2.

Remarks. For a continuous function f we see from the definitions

$$C_q(f, x, \theta) = C(f, x, \theta) \quad \text{and} \quad C_q(f, x, S) = C(f, x, S).$$

Thus, from Corollary 1, we get the following result for ordinary cluster sets.

COROLLARY 2. *If f: $H \to W$ is continuous, where W is a second countable topological space, and if $\{S\}$ is the collection of all sectors in H, then, except a σ-porous set of points $x \in R$, the set*

$$\{\theta\colon\ 0 < \theta < \pi;\ C(f, x, \theta) = \bigcup \{C(f, x, S)\colon\ S \in \{S\}\}\}$$

is residual in $(0, \pi)$.

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