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TAIL PROBABILITY ESTIMATES FOR CERTAIN RADEMACHER SUMS

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Let $\{\varepsilon_k\}$ be a *Rademacher sequence*, i.e. a sequence of identically distributed independent random variables $\varepsilon_k = \pm 1$, each with probability 1/2. It is well known that

$$\sum_{k=1}^{\infty} a_k \varepsilon_k < \infty$$

a.s. iff $\{a_k\} \in l^2$ and $\sum_{k=1}^{\infty} a_k \varepsilon_k$ is bounded iff $\{a_k\} \in l^1$. If $\{a_k\} \in l^2$,

$$\mathrm{E}\Big[\exp\Big\{\lambda\Big(\sum_{k=1}^{\infty} a_k \varepsilon_k\Big)^2\Big\}\Big] < \infty \quad \text{for all } \lambda < \infty. \tag{1}$$

Apparently there has been some speculation that a function φ could be found, that goes to infinity rapidly enough, so that

$$\mathrm{E}\Big[\varphi\Big(\sum_{k=1}^{\infty} a_k \varepsilon_k\Big)\Big] < \infty$$

would imply that $\{a_k\} \in l^1$. This cannot be done. In the following proposition we give examples of sequences $\{a_k\} \in l^2$ but not in l^1 for which the tails of the probability distributions of the random variables $\sum_{k=1}^{\infty} a_k \varepsilon_k$ go to zero very quickly.

PROPOSITION. *Let $\{\varepsilon_k\}$ be a Rademacher sequence. If $1/2 < a < 1$ and $\beta = (1-a)^{-1}$, then*

$$\frac{\big(1-o(1)\big)\,(2a-1)}{2^{\beta+1}(1-a)^2} \leqslant \frac{-\log \mathrm{P}\big[\sum_{k=1}^{\infty} \varepsilon_k k^{-a} \geqslant u\big]}{u^{\beta}(1-a)^{\beta}} \leqslant \log 2\big(1+o(1)\big) \tag{2}$$

for $u \geqslant u_0 > 0$ *for some* u_0 *sufficiently large. If* $a < 1$ *and* $\beta = (1-a)^{-1}$, *then*

$$(3) \qquad \frac{\big(1-o(1)\big)}{2^\beta} \leqslant \frac{\log\{-\log \mathrm{P}[\sum_{k=1}^{\infty} \varepsilon_k k^{-1}(\log k)^{-a} \geqslant u]\}}{u^\beta(1-a)^\beta} \leqslant 1+o(1)$$

for $u \geqslant u_0 > 0$ *for some* u_0 *sufficiently large.*

Proof. Let

$$2\sum_{k=1}^{N} k^{-a} < u \leqslant 2\sum_{k=1}^{N+1} k^{-a}.$$

We have

$$\mathrm{P}\Big[\sum_{k=1}^{\infty}\varepsilon_k k^{-a} > u\Big] \leqslant \mathrm{P}\Big[\sum_{k=1}^{N-1}\varepsilon_k k^{-a} > \sum_{k=1}^{N} k^{-a}\Big] + \mathrm{P}\Big[\sum_{k=N}^{\infty}\varepsilon_k k^{-a} > \sum_{k=1}^{N} k^{-a}\Big]$$

$$\leqslant \exp\left\{-\frac{\big(\sum_{k=1}^{N} k^{-a}\big)^2}{2\big(\sum_{k=N}^{\infty} k^{-2a}\big)}\right\} \leqslant \exp\left\{-\frac{(2a-1)N\big(1-o(1)\big)}{2(1-a)^2}\right\}$$

$$\leqslant \exp\left\{-\frac{(2a-1)}{2(1-a)^2}\left(\frac{(1-a)u}{2}\right)^\beta \big(1-o(1)\big)\right\}.$$

This holds for all u sufficiently large. The left-hand side of (2) follows. To obtain the right-hand side of (2) note that

$$\mathrm{P}\Big[\sum_{k=1}^{\infty}\varepsilon_k k^{-a} \geqslant u\Big] \geqslant \frac{1}{2}\mathrm{P}\Big[\sum_{k=1}^{M}\varepsilon_k k^{-a} \geqslant 2\sum_{k=1}^{N+1} k^{-a}\Big] \geqslant \left(\frac{1}{2}\right)^{M+1}$$

as long as

$$\sum_{k=1}^{M} k^{-a} \geqslant 2\sum_{k=1}^{N+1} k^{-a}.$$

This is the case for $M = 2^\beta N\big(1+o(1)\big)$. Therefore

$$\mathrm{P}\Big[\sum_{k=1}^{\infty}\varepsilon_k k^{-a} > u\Big] \geqslant \exp\{-(\log 2)2^\beta N\big(1+o(1)\big)\}$$

$$\geqslant \exp\{-(\log 2)\,(1-a)^\beta u^\beta\big(1+o(1)\big)\}.$$

As above, this is valid for all u sufficiently large. The right-hand side of (2) follows.

The relationships given in (3) are obtained by exactly the same analysis.

It should be evident that by taking $\{a_k\}$ even closer to a convergent sequence (but not in l^1), we can have the tails of the corresponding probability distributions going to zero, effectively, as rapidly as we please.

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