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LIE DERIVATIVES OF SECTORFORM FIELDS

BY

IVAN KOLÁŘ (BRNO)

White [9] introduced recently a useful concept of sector r-form, which generalizes the classical 1-forms to the case of the iterated tangent bundles. In the first part of the present paper, we develop a systematic theory of Lie differentiation of arbitrary sectorform fields (in [9], this question is studied only for those special sectorform fields that coincide with the classical covariant tensors). Our approach is based on a recent definition of the generalized Lie derivative and on the theory of prolongation functors [4]. Another important tool is an original idea of a T-natural transformation, which gives a simple construction of the prolongations of vector fields. Our second aim is to show that certain maps analogous to sector r-forms can be defined on the fibre bundle of all k-dimensional velocities of order r. Our construction is based on a reduction procedure dealing with the iterations of a prolongation functor of rather general type. All manifolds and maps are assumed to be infinitely differentiable.

1. T-natural transformations. Let M be the category of all manifolds and maps, and FM the category of fibred manifolds. A functor $F: M \to FM$ transforming any manifold M into a fibred manifold $p_M: FM \to M$ and any map $f: M \to N$ into a fibred manifold morphism $Ff: FM \to FN$ over f is said to be a *prolongation functor* if it satisfies two simple additional conditions of locality and regularity [4]. Any vector field ξ on M is prolonged into a vector field $F\xi$ on FM, the flow of which is the prolongation of the flow of ξ, i.e.,

$$\exp t(F\xi) = F(\exp t\xi) \tag{1}$$

(see [8]). On the other hand, ξ is a map of M into TM and one can construct $F\xi: FM \to FTM$. For $F = T$, we deduced that $T\xi = i_M \circ T\xi$, where $i_M: TTM \to TTM$ is the classical canonical involution on TTM (see [5]). From the categorical point of view, i is a natural transformation of the functor TT into itself. We are going to study a generalization of such a situation. Let $\pi_M: TM \to M$ be the bundle projection of the tangent bundle.

DEFINITION 1. A natural transformation $i\colon FT \to TF$ satisfying

$$F\pi_M = \pi_{FM} \circ i_M$$

is called a *T-natural transformation* of F if

$$F\xi = i_M \circ F\xi \tag{2}$$

for any manifold M and any vector field ξ on M.

Consider the functor T_k^r of k-dimensional velocities of order r (see [1]), i.e., $T_k^r M$ is the fibred manifold of all r-jets of $\mathbf{R}^k$ into M with source 0 and

$$(T_k^r f)(j_0^r g) = j_0^r(f \circ g)$$

for any $g\colon \mathbf{R}^k \to M$ and $f\colon M \to N$. Libermann [6] introduced a map

$$i_M\colon T_k^r TM \to TT_k^r M$$

as follows. Any $B \in TM$ is tangent to a curve $\gamma(t)$,

$$B = \frac{\partial}{\partial t}\Big|_0 \gamma(t),$$

so that any $A \in T_k^r TM$ is of the form

$$A = j_0^r\left(\frac{\partial}{\partial t}\Big|_0 \gamma(u, t)\right), \qquad u \in \mathbf{R}^k.$$

Then one defines

$$i_M(A) = \frac{\partial}{\partial t}\Big|_0 \left(j_0^r \gamma(u, t)\right) \in TT_k^r M. \tag{3}$$

PROPOSITION 1. *$i\colon T_k^r T \to TT_k^r$ is a T-natural transformation of T_k^r.*

Proof. Obviously,

$$\pi_{T_k^r M}\left(i_M(V)\right) = j_0^r \gamma(u, 0) = (T_k^r \pi_M)(A).$$

For every $f\colon M \to N$, we have

$$(TT_k^r f)\left(i_M(A)\right) = \frac{\partial}{\partial t}\Big|_0 \left(j_0^r f\left(\gamma(u, t)\right)\right) = i_N\left(T_k^r Tf(A)\right).$$

Further, let $\varphi_t(x)$ be the flow of a vector field ξ, so that

$$\xi(x) = \frac{\partial}{\partial t}\Big|_0 \varphi_t(x).$$

For any $X = j_0^r g(u) \in T_k^r M$ we have

$$(T_k^r \xi)(X) = j_0^r\left(\frac{\partial}{\partial t}\Big|_0 \varphi_t(g(u))\right).$$

On the other hand,

$$\exp t(T_k^r \xi)(X) = (T_k^r \varphi_t)(X) = j_0^r \varphi_t(g(u)).$$

Hence

$$(T_k^r \xi)(X) = \frac{\partial}{\partial t}\Big|_0 (j_0^r \varphi_t(g(u))).$$

If j: $GT \to TG$ is another T-natural transformation of a prolongation functor G, we have

$$Gi_M\colon\ GFTM \to GTFM \quad \text{and} \quad j_{FM}\colon\ GTFM \to TGFM.$$

Hence we can construct

$$k_M := j_{FM} \circ Gi_M\colon\ GFTM \to TGFM. \tag{4}$$

PROPOSITION 2. *k is a T-natural transformation of GF.*

Proof. Obviously,

$$F\pi_M = \pi_{FM} \circ i_M \quad \text{and} \quad G\pi_M = \pi_{GM} \circ j_M$$

imply

$$GF\pi_M = G\pi_{FM} \circ Gi_M = \pi_{GFM} \circ j_{FM} \circ Gi_M.$$

Further,

$$j_{FM} \circ Gi_M \circ GF\xi = j_{FM} \circ G(i_M \circ F\xi) = j_{FM} \circ G(F\xi) = GF\xi.$$

By iteration, any T-natural transformation i of F induces a T-natural transformation i^r: $F_r T \to TF_r$ of the r-th iterated functor

$$F_r = \underbrace{F \ldots F}_{r\text{ times}}.$$

If we take $F = T$, then the canonical involution i: $TT \to TT$ induces a T-natural transformation i^r: $T_r T \to TT_r$. (White has shown [9] that the group of all permutations of $r+1$ letters acts naturally on the $(r+1)$-st tangent bundle $T_{r+1} M$. Using (4) we find easily that our T-natural transformation i_M^r: $T_{r+1} M \to T_{r+1} M$ coincides with one of these permutations.)

2. Lie derivatives of sectorform fields. Some recent results suggest that the basic situation for the Lie differentiation is the following one [5]. Given two manifolds M and N, a map f: $M \to N$, a vector field ξ on M and a vector field η on N, the *Lie derivative* of f with respect to ξ and η is defined by

$$L(\xi, \eta) f := Tf \circ \xi - \eta \circ f\colon\ M \to TN. \tag{5}$$

This is a so-called *vector field along* f. Using flows [2], we can express $L(\xi, \eta)f$ as

$$L(\xi, \eta)f = \frac{\partial}{\partial t}\Big|_0 \exp(-t\eta) \circ f \circ \exp t\xi. \tag{6}$$

Given a real-valued map $f\colon M \to \boldsymbol{R}$, we have

$$Tf\colon TM \to T\boldsymbol{R} = \boldsymbol{R} \times \boldsymbol{R},$$

and the second component $\delta f\colon TM \to \boldsymbol{R}$ of Tf is called the *differential* of f. Consider a 1-form ω on a manifold M, which can be interpreted as a map $\omega\colon TM \to \boldsymbol{R}$. Then the classical Lie derivative $L_\xi \omega$ coincides with the second component of $L(T\xi, 0)\omega$, where 0 means the zero vector field on $\boldsymbol{R}$ (see [4]). Using $T\xi = i_M \circ T\xi$, we find

$$L_\xi \omega = (\delta\omega) \circ i_M \circ T\xi.$$

Consider the iterated r-th tangent bundle $T_r M$. We have r canonical projections

$$T_k \pi_{T_{r-k-1}M}\colon T_r M \to T_{r-1} M, \quad k = 0, \ldots, r-1,$$

and $T_r M$ is a vector bundle with respect to any of these projections. According to White [9], a map $A\colon (T_r M)_x \to \boldsymbol{R}$ is called a *sector r-form* at $x \in M$ if it is a linear morphism with respect to all r vector bundle structures. Let $T_r^* M \to M$ denote the fibre bundle of all sector r-forms on M (see [9]). A *sectorform field* on M is a section $A^r\colon M \to T_r^* M$ which can be interpreted as a map $A^r\colon T_r M \to \boldsymbol{R}$. If ξ is a vector field on M, then

$$L(T_r \xi, 0)A^r\colon T_r M \to T\boldsymbol{R}. \tag{7}$$

DEFINITION 2. The *Lie derivative* $L_\xi A^r$ is the second component of (7).

By Proposition 2 we obtain

PROPOSITION 3. *We have* $L_\xi A^r = (\delta A^r) \circ i_M^r \circ T_r \xi$.

For any subcomplex K of an $(r-1)$-dimensional simplex, White constructs a fibre bundle $T_r[M; K]$ over M and a canonical projection $T_r M \to T_r[M; K]$. A sector r-form is called a *sector K-form* if it is projectable with respect to the latter projection. Since this construction has a functorial character, it follows from (6) and (7) that the Lie derivative of a K-sectorform field is a K-sectorform field. In particular, if K is the subcomplex of all vertices of an $(r-1)$-dimensional simplex, then $T_r[M; K]$ coincides with the Whitney sum $\bigoplus^r TM$, and a K-sectorform field is a classical r-times covariant tensor field on M. In this case, (6) and (7) imply that we get the classical Lie derivative.

3. Generalized sectorforms. Let F be a prolongation functor with values in the category $\boldsymbol{VB}$ of vector bundles. Assume further that F satisfies

(i) (linearity axiom) for any vector bundle q: $E \to M$, Fq: $FE \to FM$ is also a vector bundle, and for any linear morphism

$$f\colon (E \to M) \to (D \to N),$$

Ff: $(FE \to FM) \to (FD \to FN)$ is a linear morphism;

(ii) (Pradines' axiom [7]) if $Y \to X$ is a submersion, then FY is a submersion over $Y \oplus FX$ with respect to the pullback map.

As a consequence of Pradines' axiom, if x^i, y^p are some fibre coordinates on Y, and x^i, z^α are some fibre coordinates on FX, then there are some fibre coordinates on FY of the form x^i, y^p, z^α, w^λ.

Consider the r-th iteration F_r of F. Then we have r canonical projections

$$F_k p_{F_{r-k-1}M}\colon F_r M \to F_{r-1} M, \qquad k = 0, \ldots, r-1,$$

and $F_r M$ is a vector bundle with respect to any of these projections. A map A: $(F_r M)_x \to \boldsymbol{R}$ is called an (F, r)-*form* if A is a linear morphism with respect to all vector bundle structures. We are going to deduce the coordinate expression of an (F, r)-form. Given some local coordinates x^i on M, let $X_1^{p_1}$ be some additional fibre coordinates on FM. On $F_2 M$, we denote by x^i, $X_{10}^{p_1}$ the coordinates induced from $F_1 M$ by p_{FM}: $F_2 M \to FM$. According to Pradines' axiom, the additional coordinates on $F_2 M$ are $X_{01}^{p_1}$, $X_{11}^{p_2}$, where the superscript p_2 corresponds to the fibre dimension of $F_2 M \to FM \oplus FM$. In general, consider a sequence $\gamma = (\gamma_1, \ldots, \gamma_r)$, where $\gamma_i \in \{0, 1\}$, not all γ_i zero, and set $|\gamma|$ to be the number of 1's in the sequence, and $e = (1, 1, \ldots, 1)$. By Pradines' axiom, we have a pullback projection

$$F_r M \to F_{r-1} M \oplus_{F_{r-2}M} F_{r-1} M. \tag{8}$$

On $F_r M$, we define local coordinates x^i, $X_\gamma^{p|\gamma|}$ for all γ as follows. If $\gamma_r = 0$, then $X_\gamma^{p|\gamma|}$ is induced by the corresponding coordinate on the first factor of (8); if $\gamma_{r-1} = 0$, then $X_\gamma^{p|\gamma|}$ is induced by the corresponding coordinate on the second factor of (8) (if both $\gamma_{r-1} = 0 = \gamma_r$, we are on the basis $F_{r-2} M$ of the Whitney sum (8)), and $X_e^{p_r}$ are the additional coordinates in the fibres of (8). According to White [9], if

$$\gamma' = (\gamma'_1, \ldots, \gamma'_r) \quad \text{and} \quad \gamma'' = (\gamma''_1, \ldots, \gamma''_r)$$

satisfy $\gamma'_i \gamma''_i = 0$ for all i $(1 \leqslant i \leqslant r)$, then the join $\gamma = \gamma' \cup \gamma''$ is defined by $\gamma_i = \gamma'_i + \gamma''_i$. Any expression of the form

$$\gamma_1 \cup \ldots \cup \gamma_k = e$$

is called a *join decomposition* of e. By an induction procedure quite similar to the proof of Theorem 3.2 of [9], one proves

PROPOSITION 4. *A map* $A\colon (F_r M)_x \to \boldsymbol{R}$ *is an* (F, r)*-form if and only if it has the coordinate expression*

$$\sum_{\gamma_1 \cup \ldots \cup \gamma_k = e} a^{\gamma_1 \ldots \gamma_k}_{p_{i_1} \ldots p_{i_k}} X^{p_{i_1}}_{\gamma_1} \ldots X^{p_{i_k}}_{\gamma_k}, \tag{9}$$

where $i_j = |\gamma_j|$ *and the sum is taken over all distinct join decompositions of* e.

(9) implies that the set of all (F, r)-forms at $x \in M$ is a vector space. Hence the space of all (F, r)-forms on M is a vector bundle $F^*_r M \to M$. For any map $f\colon N \to M$, $f(y) = x$,

$$(F_r f)_y\colon (F_r N)_y \to (F_r M)_x$$

is a linear morphism with respect to all r vector bundle structures, so that $A \circ (F_r f)_y$ is an (F, r)-form at $y \in N$ for any $A \in (F_r M)_x$. By (9) and the linearity axiom, the induced map

$$(F^*_r f)_y\colon (F^*_r M)_x \to (F^*_r N)_y$$

is linear. Consider the dual vector space

$$(F'_r M)_x := (F^*_r M)^*_x$$

and the dual map

$$(F'_r f)_y := (F^*_r f)^*_y\colon (F'_r N)_y \to (F'_r M)_x.$$

In this way, we obtain a prolongation functor $F'_r\colon \boldsymbol{M} \to \boldsymbol{VB}$ (the exact proof follows from the basic facts of the theory of prolongation cofunctors explained in [3]).

Having a vector field ξ on M, the *Lie derivative* $L_\xi A^r$ *of an* (F, r)*-form field* $A^r\colon F_r M \to \boldsymbol{R}$ is defined as the second component of $L(F_r \xi, 0) A^r$.

PROPOSITION 5. $L_\xi A^r$ *is an* (F, r)*-form field for any* ξ.

Proof. The prolonged flow $F_r(\exp t\xi)$ transforms any (F, r)-form into an (F, r)-form. By (9), the derivative of a one-parameter family of (F, r)-forms is also an (F, r)-form. Then (6) implies our assertion.

If we have a T-natural transformation $i\colon FT \to TF$ and we construct the induced T-natural transformation $i^r\colon F_r T \to TF_r$, then we deduce in the same way as in Section 2 that

$$L_\xi A^r = (\delta A^r) \circ i^r_M \circ F_r \xi. \tag{10}$$

4. T^r_k-forms. We have

$$T^1_k M = \bigoplus_k TM \quad \text{and} \quad T^1_k f = \bigoplus_k Tf,$$

so that the values of the functor T^1_k are in $\boldsymbol{VB}$. One verifies easily that T^1_k

satisfies conditions (i) and (ii) of Section 3. To study T_k^r by means of $(T_k^1)_r$, we introduce a map

$$h^r\colon\ T_k^r M \to (T_k^1)_r M$$

by the following induction. Let $t_v\colon \mathbf{R}^k \to \mathbf{R}^k$ be the translation $u \mapsto u+v$, $u \in \mathbf{R}^k$. Any $X \in T_k^r M$ is of the form $X = j_0^r g(u)$. Then we set

$$\tilde{g}(v) = j_0^{r-1}(g \circ t_v) \in T_k^{r-1} M \quad \text{and} \quad h^r(X) = j_0^1 h^{r-1}(\tilde{g}(u)), \quad h^1 = \mathrm{id}.$$

Using local coordinates, we find directly that h^r is injective, so that we can write $T_k^r M \subset (T_k^1)_r M$. Moreover, $T_k^r f$ is the restriction of $(T_k^1)_r f$ to $T_k^r M$ for any $f\colon M \to N$. We define a T_k^r-form at $x \in M$ to be the restriction of a (T_k^1, r)-form $((T_k^1)_r M)_x \to \mathbf{R}$ to $(T_k^r M)_x$. Any local coordinates x^i on M induce fibre coordinates x^i, X_α^i on $T_k^r M$, where α are multi-indices $|\alpha| \leqslant r$ corresponding to the partial derivatives on $\mathbf{R}^k$. Comparing the notation, we deduce from (9)

PROPOSITION 6. *A map $(T_k^r M)_x \to \mathbf{R}$ is a T_k^r-form if and only if it has the coordinate expression*

$$\sum_{|\alpha_1|+\ldots+|\alpha_s|=r} a_{i_1 \ldots i_s}^{\alpha_1 \ldots \alpha_s} X_{\alpha_1}^{i_1} \ldots X_{\alpha_s}^{i_s}, \tag{11}$$

where the sum is taken over all distinct sequences of multi-indices satisfying $|\alpha_1| + \ldots + |\alpha_s| = r$.

From Section 3 it follows that the set $(T_k^r)_x^* M$ of all T_k^r-forms at $x \in M$ is a vector space and

$$(T_k^r)^* M = \bigcup_{x \in M} (T_k^r)_x^* M$$

is a vector bundle over M. For every $f\colon N \to M$, $f(y) = x$, we obtain an induced linear map $(T_k^r)_x^* M \to (T_k^r)_y^* N$ and the dual constructions lead to a prolongation functor $T_k^{r'}\colon \boldsymbol{M} \to \boldsymbol{VB}$.

The *Lie derivative $L_\xi A^r$ of a T_k^r-form field A^r*: $T_k^r M \to \mathbf{R}$ is defined as the second component of $L(T_k^r \xi, 0) A^r$. Using flows, we deduce that $L_\xi A^r$ is also a T_k^r-form field. In the same way as in Section 3, we find

$$L_\xi A^r = (\delta A^r) \circ i_M \circ T_k^r \xi,$$

where i_M is the T-natural transformation of Proposition 1.

REFERENCES

[1] C. Ehresmann, *Les prolongements d'une variété différentiable. I*, C. R. Acad. Sci. Paris 233 (1951), pp. 598–600.

[2] J. Janyška and I. Kolář, *Lie derivatives on vector bundles*, pp. 111–116 in: *Proceedings of*

the Conference on Differential Geometry and its Applications, Universita Karlova Praha, 1982.

[3] T. Klein, *Connections on higher order tangent bundles*, Časopis Pěst. Mat. 106 (1981), pp. 414–421.

[4] I. Kolář, *Structure morphisms of prolongation functors*, Math. Slovaca 30 (1980), pp. 83–93.

[5] – *On the second tangent bundle and generalized Lie derivatives*, Tensor (N. S.) 38 (1982), pp. 98–102.

[6] P. Libermann, *Parallélismes*, J. Differential Geom. 8 (1973), pp. 511–539.

[7] J. Pradines, *Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux*, C. R. Acad. Sci. Paris, Sér. A, 264 (1967), pp. 245–248.

[8] S. A. Salvioli, *On the theory of geometric objects*, J. Differential Geom. 7 (1972), pp. 257–278.

[9] J. E. White, *The Methods of Iterated Tangents with Applications to Local Riemannian Geometry*, Pitman Press, 1982.

INSTITUTE OF MATHEMATICS OF THE ČSAV
BRANCH BRNO
BRNO

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