COLLOQUIUM MATHEMATICUM

VOL. XLVIII 1984 FASC. 1

EXTREME SPECTRAL STATES AND MULTIPLICATIVE EXTENSIONS IN BANACH ALGEBRAS

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1. Introduction. Let A be a *complex unital Banach algebra*, i.e. a Banach algebra with unit satisfying $\|1\| = 1$. Following Bonsall and Duncan [1] we call a linear functional f on A a *spectral state* if

$$f(1) = 1 \quad \text{and} \quad |f(x)| \leqslant \varrho(x) \quad (x \in A),$$

where $\varrho(x)$ stands for the spectral radius of x. The set $\Omega(A)$ of all spectral states of A is readily seen to be a convex and weak* compact subset of the unit sphere in the dual space A^* of A. It may happen that $\Omega(A)$ is empty (see [1], p. 115); this never occurs in commutative algebras, since each non-zero multiplicative functional is obviously a spectral state.

By using well-known ideas of the functional analysis we show that if A is commutative, then every extreme point of $\Omega(A)$ is a non-zero multiplicative functional on A and that the weak* closure of ext $\Omega(A)$ is equal to the Šilov boundary of A. Next we characterize those complex unital Banach algebras A such that for each subalgebra B of A and each element f in ext $\Omega(B)$ there exists an extension $\tilde{f} \in \Omega(A)$ of f. It turns out that the algebras with the above property are close to the commutative ones. This leads to a more general statement of the Šilov extension theorem.

Throughout this paper all Banach algebras are complex and unital. In saying that B is a subalgebra of A it is always assumed that B is closed and contains the unit of A. If A is commutative, then we write $\Delta(A)$ for the non-empty weak* compact set of all non-zero multiplicative functionals on A, equipped with the (relative) weak* topology. We denote by $\hat{A}$ the image of A under the Gelfand homomorphism $x \to \hat{x}$, where $\hat{x}(h) = h(x)$, $h \in \Delta(A)$. Then $\hat{A}$ is a point-separating subalgebra of the Banach algebra $C(\Delta(A))$ of all complex-valued continuous functions on $\Delta(A)$, and the sup norm $\|\hat{x}\|$, of $\hat{x} \in \hat{A}$ is equal to the spectral radius of x.

If E is a subset of the dual space of a normed space, then $\bar{E}$ will always denote the weak* closure of E.

2. Spectral states on commutative algebras.

LEMMA 1. *If the unital Banach algebra A is commutative, then $\Omega(A)$ coincides with the set all functionals of the form*

$$f(x) = \int \hat{x} d\mu, \tag{1}$$

where μ is a regular Borel probability measure on $\Delta(A)$.

Proof. Every functional of the form (1) is clearly in $\Omega(A)$. Conversely, let $f \in \Omega(A)$. Define

$$\hat{f}(\hat{x}) = f(x) \quad (x \in A). \tag{2}$$

Since $|\hat{f}(\hat{x})| \leqslant \varrho(x)$ and $\hat{f}(1) = 1$, $\hat{f}$ is a well-defined functional of norm one on $\hat{A}$. By the Hahn-Banach and Riesz theorems there exists a regular Borel probability measure satisfying (1).

As a consequence of Lemma 1 we have the following proposition:

PROPOSITION 1. *If A is a commutative unital Banach algebra, then $\Omega(A)$ is equal to the weak* closed convex hull of $\Delta(A)$.*

Proof. By, e.g., [4], Proposition 1.2, every functional of the form (1) is in $\overline{\text{co}\, \Delta(A)}$. On the other hand, $\Omega(A) \supset \Delta(A)$, so $\Omega(A) = \overline{\text{co}\, \Delta(A)}$.

By the Milman theorem (see, e.g., [4], p. 9), Proposition 1 is equivalent to the following

PROPOSITION 2. *If A is a commutative unital Banach algebra, then* $\text{ext}\, \Omega(A) \subset \Delta(A)$.

We recall that the *Šilov boundary* of a commutative Banach algebra is defined as the smallest closed subset of $\Delta(A)$ on which the absolute value of each function $\hat{x} \in \hat{A}$ attains its maximum. The following observation, which we use in the sequel, is of independent interest.

PROPOSITION 3. *If A is a commutative unital Banach algebra, then the closure of* ext $\Omega(A)$ *is equal to the Šilov boundary of A.*

Proof. For each $f \in \Omega(A)$ let f be defined by (2). It is clear that $f \to \hat{f}$ is a one-to-one affine homeomorphism of $\Omega(A)$ onto the weak* compact convex set $S(\hat{A}) = \{F \in (A, \|\cdot\|_\infty)^*\colon \|F\| = 1 = F(1)\}$. It follows immediately from [4], p. 40, that the Šilov boundary of A is carried via this mapping onto $\overline{\text{ext}\, S(\hat{A})}$. Since the latter is the image of $\overline{\text{ext}\, \Omega(A)}$, the proof is complete.

It is now easy to see that the inclusion in Proposition 2 is in general proper. For example, take the disk algebra $\mathscr{A}(D)$. We have

$$\overline{\text{ext}\, \Omega(\mathscr{A}(D))} = \partial D \subsetneq D = \Delta(\mathscr{A}(D))$$

(for the first equality see, e.g., [4], p. 54).

3. Spectral states and positive functionals on commutative star algebras. Now let A be a commutative unital Banach algebra with an involution $*$. We recall that a linear functional f on A is called *positive*, $f \geqslant 0$, if $f(x^* x) \geqslant 0$ for every $x \in A$. It is well known (see, e.g., [5], p. 284) that each positive functional is bounded with $\|f\| = f(1)$ and satisfies $f(x^*) = \overline{f(x)}$ as well as $|f(x)| \leqslant f(1)\, \varrho(x)$ for every $x \in A$. The set $P(A)$ of all positive functionals of norm one on A is convex and weak* compact. It is immediate that each element of $P(A)$ is a spectral state.

It may be interesting to note that the equality $P(A) = \Omega(A)$ occurs if and only if A is symmetric or, equivalently, if and only if $\Delta(A) \subset P(A)$. Indeed, the latter condition and Proposition 1 imply $\Omega(A) = \overline{\operatorname{co} \Delta(A)} \subset P(A)$. Conversely, if $P(A) = \Omega(A)$, then clearly $\Delta(A) \subset P(A)$ since every non-zero multiplicative functional is in $\Omega(A)$.

By Lemma 1, every element f of $P(A)$ can be represented by a regular Borel probability measure on $\Delta(A)$. However, in the present case a sharper result is possible. Namely, as Bucy and Maltese showed in [2], ext $P(A)$ coincides with the weak* compact set $P(A) \cap \Delta(A)$ (see also [5], p. 286, for a simpler proof of this fact). Hence it follows from the Krein-Milman theorem that there exists a regular Borel probability measure μ on $P(A) \cap \Delta(A)$ such that

$$f(x) = \int_{P(A) \cap \Delta(A)} \hat{x}\, d\mu \qquad (x \in A).$$

Moreover, such μ is unique, since the algebra $\hat{A}$ restricted to $P(A) \cap \Delta(A)$ is dense in $C\big(\Delta(A) \cap P(A)\big)$ by the Stone-Weierstrass theorem.

4. Extensions of spectral states. If B is a subalgebra of a unital Banach algebra A and if $f \in \Omega(B)$, then we call a functional $\tilde{f}$ on A a *spectral extension* of f if $\tilde{f}|_B = f$ and $\tilde{f} \in \Omega(A)$.

The proof of the following lemma is analogous to that of Maltese [3].

LEMMA 2. *Let A be a unital Banach algebra and B a subalgebra of A. If f is in* ext $\Omega(B)$, *then f can be extended to an element of* ext $\Omega(A)$ *if and only if f has a spectral extension.*

Proof. The "only if" part is trivial. Suppose that f has a spectral extension. It follows that the convex and weak* compact set X defined by

$$X = \{g \in \Omega(A) \colon\ g|_B = f\}$$

is non-empty; therefore, ext $X \neq \emptyset$ by the Krein-Milman theorem. We will show that

$$\operatorname{ext} X = X \cap \operatorname{ext} \Omega(A).$$

It suffices to prove that X is an extreme subset of $\Omega(A)$. Suppose $g \in X$ and g

$=\alpha g_1+(1-\alpha)g_2$ with $g_i\in\Omega(A)$, $i=1, 2$, and $0<\alpha<1$. By restricting the last equality to B we get

$$f=g|_B=\alpha g_1|_B+(1-\alpha)g_2|_B.$$

Clearly, $g_i|_B\in\Omega(B)$. By assumption, f is in ext $\Omega(B)$ and, consequently, $g_1|_B = g_2|_B = f$, which means that $g_i\in X$, $i=1, 2$, as desired.

LEMMA 3. *Let A be a unital Banach algebra and B a subalgebra of A. Suppose that every element of* ext $\Omega(B)$ *has a spectral extension. Then every element of* $\overline{\text{ext}\ \Omega(B)}$ *has an extension to an element of* $\overline{\text{ext}\ \Omega(A)}$.

Proof. Fix $f\in\overline{\text{ext}\ \Omega(B)}$ and let $\{f_\alpha\}$ be a net in ext $\Omega(B)$ such that $f_\alpha\to f$. By assumption and Lemma 2, each f_α extends to $\tilde{f}_\alpha\in\text{ext}\ \Omega(A)$. By compactness, we can find a subnet $\{\tilde{f}_\beta\}$ of $\{\tilde{f}_\alpha\}$ weak* convergent to some g, the desired extension of f.

The theorem below explains the connection between the existence of spectral extensions and spectral properties of the algebra in question.

THEOREM 1. *Let A be a unital Banach algebra. The following conditions are equivalent:*

(i) *for every subalgebra $B\subset A$ and every $f\in\Omega(B)$ there exists a spectral extension of f;*

(ii) *for every subalgebra $B\subset A$ and every $f\in$ ext $\Omega(B)$ there exists a spectral extension of f;*

(iii) *for each $x\in A$ there exists a commutative subalgebra $B\subset A$ containing x and such that for each $f\in$ ext $\Omega(B)$ there exists a spectral extension of f;*

(iv) *the spectral radius is a seminorm on A.*

Proof. The implication (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are trivial. That (iv) implies (i) follows from the definition of a spectral state and from the Hahn-Banach extension theorem. It remains to show that (iii) implies (iv). For this purpose it suffices to prove that the spectral radius is subadditive on A. Fix $x, y\in A$ and let B be a commutative subalgebra of A containing $x+y$ and such that each element of ext $\Omega(B)$ has a spectral extension. By Lemma 3, each $h\in\overline{\text{ext}\ \Omega(B)}$ can be extended to a spectral state of A. By Proposition 3 we have

$$\begin{aligned}\varrho(x+y) &= \sup\{|h(x+y)|:\ h\in\overline{\text{ext}\ \Omega(B)}\}\leqslant \sup\{|f(x+y)|: f\in\Omega(A)\}\\ &\leqslant \sup\{|f(x)|: f\in\Omega(A)\}+\sup\{|f(y)|: f\in\Omega(A)\}\\ &\leqslant \varrho(x)+\varrho(y)\end{aligned}$$

and the proof is complete.

Denote by Rad A the radical of A. As J. Zemanek showed in [6], the commutativity of $A/\text{Rad}\,A$ is equivalent to the subadditivity of the spectral radius, thus to the condition (iv) of Theorem 1. (We mention at this point that the subadditivity of the spectral radius is an elementary consequence of the commutativity of $A/\mathrm{Rad}\,A$, unlike the inverse implication, the proof of which demands more advanced techniques.)

The following statement contains the well-known Šilov extension theorem.

THEOREM 2. *Let A be a unital Banach algebra. The following conditions are equivalent:*

(i) *$A/\mathrm{Rad}\,A$ is commutative;*

(ii) *for every commutative subalgebra B of A and every element f in the Šilov boundary $\Gamma(B)$ of B there exists a multiplicative linear extension of f to A.*

Proof. Suppose (i) holds. Obviously, $f(\mathrm{Rad}\,A) = 0$ for every $f \in \Omega(A)$. The canonical mapping of $\Omega(A/\mathrm{Rad}\,A)$ onto $\Omega(A)$ is easily seen to be an affine weak* homeomorphism. Moreover, multiplicative functionals on $A/\mathrm{Rad}\,A$ are carried onto multiplicative functionals on A. Let B be a commutative subalgebra of A. From (i) it follows easily that ϱ is a seminorm, therefore by Theorem 1 every element of ext $\Omega(B)$ admits a spectral extension. This, together with Lemma 3 and Proposition 3, shows that each $f \in \Gamma(B)$ extends to an element $\tilde{f} \in \overline{\mathrm{ext}\,\Omega(A)}$. Since $\tilde{f}$ can be viewed as an element of $\overline{\mathrm{ext}\,\Omega(A/\mathrm{Rad}\,A)}$, from the commutativity of $A/\mathrm{Rad}\,A$ we conclude by Proposition 2, that $\tilde{f}$ is multiplicative.

Conversely, suppose (ii) holds. It is easy to see that then the condition (iii) of Theorem 1 is satisfied. We now apply this theorem and Zemanek's result [6] to obtain the commutativity of $A/\mathrm{Rad}\,A$.

Remark. It has been shown by Żelazko ([7], Corollary 3) that given a commutative unital Banach algebra A and a commutative norm- and unit-preserving Banach superalgebra C of A, one can extend every functional $f \in \Gamma(A)$ to a member of $\Gamma(C)$. It is easily seen that this result can be obtained also by the direct application of our Lemma 3 together with Proposition 3.

We are indebted to Anzelm Iwanik for his remarks and suggestions during the preparation of this paper.

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Reçu par la Rédaction le 15. 8. 1979