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CHARACTERIZATION OF AMENABLE GROUPS AND THE LITTLEWOOD FUNCTIONS ON FREE GROUPS

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1. Introduction. In his paper [5] Varopoulos introduced so-called Littlewood tensors. We shall consider properties of such functions on discrete groups. A new characterization of amenable discrete groups is given. Furthermore, we describe the behaviour of Littlewood functions on free groups.

2. Preliminaries. We consider the space $T_p(G)$, $1 \leqslant p < \infty$, of functions on a discrete group G which can be represented as the sum of two functions on $G \times G$,

$$a(x^{-1}y) = a_1(x, y) + a_2(x, y),$$

so that the following conditions are satisfied ($x, y \in G$):

(i) $a_1(x, y)\, a_2(x, y) = 0$;

(ii) there exists a constant $c > 0$ such that

$$\sup_{x\in G}\Big(\sum_{y\in G} |a_1(x, y)|^p\Big)^{1/p} \leqslant c, \quad \sup_{y\in G}\Big(\sum_{x\in G} |a_2(x, y)|^p\Big)^{1/p} \leqslant c.$$

The infimum of such constants $c > 0$ taken over all possible representations of the function a will be denoted by $|a|_{t_p}$, and, in fact, $|\cdot|_{t_p}$ is a complete norm on $T_p(G)$. Let us define the second norm on the space $T_p(G)$ as follows:

$$\|a\|_{t_p} = \sup\Big\{\Big((\max_{j=1,2} |A_j|)^{-1} \sum_{x\in A_1} \sum_{y\in A_2} |a(x^{-1}y)|^p\Big)^{1/p}:$$
$$A_1, A_2 \text{ are nonempty finite subsets of } G\Big\},$$

$|A_j|$ denotes the cardinality of A_j.

Remark. It suffices to take the supremum in the above formula over all finite sets A_1, A_2 of equal cardinality. It is easy to see that, for $1 \leqslant p$, $T_p(G)$ with each of the norms $|\cdot|_{t_p}$ and $\|\cdot\|_{t_p}$ is a Banach space and

$$|a|_{t_p} \leqslant \|a\|_{t_p} \leqslant 2^{1/p} |a|_{t_p}.$$

The first inequality can be obtained in a similar manner as in [5], Lemma 5.1, for $p = 2$; the other one follows from (i) and (ii).

3. Characterization of amenable groups. In the sequel the following descriptions of discrete amenable groups will be useful:

FØLNER–LEPTIN CONDITION. *A discrete group G is amenable if and only if for every positive number ε and every finite subset K of G there exists a finite subset U of G such that*

$$|UK| \leqslant (1+\varepsilon)|U|, \quad \textit{where } UK = \{xy \in G\colon x \in U,\ y \in K\}.$$

HULANICKI CONDITION. *A discrete group G is amenable if and only if for every positive function $f \in l^1(G)$ the two norms*

$$|f|_1 = \sum_{x \in G} |f(x)| \quad \textit{and} \quad |f|_{VN} = \sup\{|g*f|_2\colon |g|_2 = 1\}$$

are equivalent. Here

$$g*f(x) = \sum_{y \in G} g(y) f(y^{-1}x).$$

Now we are in a position to establish the main

THEOREM 1. *A discrete group G is amenable if and only if for all (or, equivalently, for some) finite $p \geqslant 1$*

$$T_p(G) = l^p(G).$$

Proof. Necessity. For $a \in l^p(G)$, $1 \leqslant p < \infty$, we have the obvious representation $a = a_1 + 0$, where $a_1(x, y) = a(x^{-1}y)$. This gives the inclusion $l^p(G) \subset T_p(G)$ for each discrete group G. Let now $a \in T_p(G)$ for some finite $p \geqslant 1$. Let K be a finite set in G and $\varepsilon > 0$. Taking a finite set U for K and ε (see the Følner–Leptin condition) we obtain

$$\|a\|_{t_p}^p \geqslant |U| \cdot |UK|^{-1} \sum_{z \in K} |a(z)|^p \geqslant \frac{1}{1+\varepsilon} \sum_{z \in K} |a(z)|^p,$$

and therefore $a \in l^p(G)$.

Sufficiency. Let $l^p(G) = T_p(G)$ for some finite $p \geqslant 1$. We always have $\|a\|_{t_p} \leqslant 2^{1/p}|a|_p$, so by the Banach Closed Graph Theorem there exists a constant $d > 0$ such that $d|a|_p \leqslant \|a\|_{t_p}$ for all $a \in l^p(G)$. The Hulanicki condition can be now easily checked.

4. Radial functions on free groups. Let $F_r = G$ be the free group on r free generators, r finite. We denote by $|x|$ the length of the word x: $|x| = 0$ if and only if x is the identity of G, $E_n = \{x \in G\colon |x| = n\}$, $n = 0, 1, 2, \ldots$, and χ_n is the characteristic function of E_n. One can verify that

$$|E_n| = 2r(2r-1)^{n-1}, \quad n = 1, 2, \ldots,$$

and

$$|\chi_n|_p = |E_n|^{1/p}.$$

The function f on G is called *radial* if $|x| = |y|$ implies $f(x) = f(y)$.

PROPOSITION 1. *For every positive integer n*

$$(2r)^{-1/p}|\chi_n|_{2p} \leqslant \|\chi_n\|_{t_p} \leqslant 3^{1/p}|\chi_n|_{2p}.$$

Proof. Let $s, n \in N$, $s > n$. Taking $A_1 = A_2 = E_s \cup E_{s+1}$ one can see that the first inequality holds. In order to see the other one we define

$$a_1(x, y) = \begin{cases} \chi_n(x^{-1}y) & \text{if } |x| \geqslant |y|, \\ 0 & \text{otherwise} \end{cases}$$

and

$$a_2(x, y) = \chi_n(x^{-1}y) - a_1(x, y).$$

Such a representation of χ_n gives

$$|\chi_n|_{t_p}^p \leqslant \begin{cases} (2r-2)^{-1}(2r-1)^{(n+2)/2} & \text{for even } n, \\ (2r-2)^{-1}(2r-1)^{(n+1)/2} & \text{for odd } n, \end{cases}$$

so the second inequality holds.

We get immediately

COROLLARY. $\|\chi_1\|_{t_p} = 2^{1/p}$.

PROPOSITION 2. *For every sequence (a_n) of complex numbers and a positive integer k the norm*

$$\|\sum_{n=0}^{k} a_n \chi_n\|_{t_p}$$

is equivalent to

$$(\sum_{n=0}^{k} |a_n|^p |\chi_n|_2)^{1/p}.$$

Proof. Taking $s > k$ and $V = E_s \cup E_{s+1}$ we get

$$\|\sum_{n=0}^{k} a_n \chi_n\|_{t_p} \geqslant (|V|^{-1} \sum_{n=0}^{k} |a_n|^p \sum_{x,y \in G} \chi_n(x^{-1}y))^{1/p}$$

$$\geqslant (2r)^{-1/2p}(\sum_{n=0}^{k} |a_n|^p |\chi_n|_2)^{1/p}.$$

Let now A and B be finite sets in G, $|A| \leqslant |B| = m$, I_A and I_B their characteristic functions; then

$$\frac{1}{m}\langle I_A * |\sum_{n=0}^{k} a_n \chi_n|^p, I_B\rangle = \frac{1}{m}\sum_{n=0}^{k} \langle I_A * \chi_n, I_B\rangle |a_n|^p$$

$$\leqslant \sum_{n=0}^{k} |a_n|^p \|\chi_n\|_{t_p}^p \leqslant c^p \sum_{n=0}^{k} |a_n|^p |\chi_n|_2$$

for some constant c.

Let us denote by $T_p^{\mathrm{rad}}(G)$ the subspace of radial functions in $T_p(G)$. The above propositions lead us to

THEOREM 2. *The space $T_p^{\mathrm{rad}}(G)$ is topologically isomorphic to the space $l^p(N, \nu)$ for the measure ν on N defined by*

$$\nu = \sum_{n=0}^{\infty} |\chi_n|_2 \, \delta_n,$$

where δ_n is the unit measure concentrated in n.

Define a linear projection M onto radial functions on G by

$$Mf(x) = |E_n|^{-1} \sum_{|y|=n} f(y) \quad \text{whenever } |x| = n.$$

THEOREM 3. *M is a bounded operator from $T_p(G)$ onto $T_p^{\mathrm{rad}}(G)$ for every p, $1 \leqslant p < \infty$.*

Proof. Let $f \in T_p(G)$ for fixed p. Observe that

$$Mf = \sum_{n=0}^{\infty} u_n \chi_n,$$

where u_n is the value of Mf on E_n. Theorem 2 gives the inequality

$$\|Mf\|_{t_p} \leqslant c\Big(\sum_{n=0}^{\infty} |u_n|^p |E_n|^{1/2}\Big)^{1/p}.$$

If

$$f_m = \sum_{n=0}^{m} f\chi_n,$$

m fixed, then taking $s > m$ and $V = E_s \cup E_{s+1}$ we get

$$\|f\|_{t_p}^p \geqslant \sum_{n=0}^{m} |V|^{-1} \sum_{x,y\in V} \chi_n(x^{-1}y)\,|f_m(x^{-1}y)|^p = \sum_{n=0}^{m} L_n.$$

We estimate the L_n's. For even n, $\chi_n(x^{-1}y) \neq 0$ only if $|x| = |y|$, and for odd n, $\chi_n(x^{-1}y) \neq 0$ only if $\big||x|-|y|\big| = 1$. Therefore

$$L_n \geqslant \frac{2r-2}{2r}(2r-1)^{-n/2} \sum_{|w|=n} |f_m(w)|^p.$$

Since m was chosen arbitrarily, we obtain

$$\|f\|_{t_p}^p \geqslant (2r)^{-1/2} \sum_{n=0}^{\infty} |E_n|^{-1/2} \sum_{|w|=n} |f_m(w)|^p,$$

and by Hölder's inequality we have

$$\|Mf\|_{l_p} \leqslant c\|f\|_{l_p}$$

with the constant c depending only on r and p.

Remark. Recently Marek Bożejko has applied Varopoulos' result on Littlewood functions to give a different characterization of amenable groups (see [1]).

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