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SOME MODEL THEORY OF SIMPLE ALGEBRAIC GROUPS OVER ALGEBRAICALLY CLOSED FIELDS

BY

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Let G be a simple algebraic group over an algebraically closed field K. It was proved in [5] that the group-language theory of G is categorical in uncountable powers, i.e. for every uncountable power μ there is a unique, up to isomorphisms, group G' of cardinality μ, which satisfies the same group-theoretical conditions as G. It is worth noting that for the proof no special information on the structure of G is needed.

One can easily observe that the theorem implies that every G', satisfying the same group-theoretical axioms as G, can be equipped with the algebraic-geometrical structure of the same kind as that of G. In some sense this means that the algebraic-geometrical structure on such groups is indeed defined by the abstract group structure.

Another, more exact version of the last fact is given by the known theorem of Borel and Tits [1], which states, in particular, that every (abstract) automorphism of G is a composition of a rational automorphism of G and of an automorphism induced by an automorphism of the field K.

The known proofs of the theorem use a deep structural theory of simple algebraic groups. Since the analogous theorem of [5] was proved by different and easier methods, the same kind of proof can be expected for the theorem cited above. Such a proof of the Borel–Tits theorem is given in the paper.

One of our purposes is also to show how model-theoretical methods can work in the theory of algebraic groups and, more generally, in algebraic geometry. There are two basic facts which link the general algebraic geometry with model theory. First, since the theory of algebraically closed fields of a given characteristic is categorical in uncountable powers, all structures definable in an algebraically closed field, e.g., structures of algebraic geometry, are ω-stable of finite Morley rank. Moreover, the Morley rank is a good analogue of the algebraic-geometrical dimension and in most cases coincides with it. The second fact is the theorem of A. Tarski, which states that every relation definable in an algebraically closed field is a Boolean combination of polynomial equations.

We suppose the reader is familiar with main facts of the theory of categoricity (see, e.g., [4]). We also use some standard facts of the theory of algebraic groups, which can be found in [2].

1. Connected solvable groups of finite Morley rank. In this section, B is a group whose theory is ω-stable and of finite Morley rank. "Definable" always means "definable with parameters".

A group B is called *m-connected* (*model connected*) if B does not contain any definable subgroup H of finite index in B.

Let H be an arbitrary subgroup of B. The minimal definable subgroup $\tilde{H} \supseteq H$ is called the *m-closure* (*model-closure*) of H in B.

LEMMA 1. *If H is solvable* (*nilpotent*) *of class n, then so is $\tilde{H}$.*

Proof: For $n = 1$ this was proved in [4], Lemma 7. Proceeding by induction and using the same method we get the lemma for all n.

LEMMA 2. *Let B be an m-connected solvable group with trivial center. Let V be a minimal normal definable subgroup of B and let the factor group $T = B/C(V)$ be commutative* ($C(V)$ *denotes the centralizer of V in B*). *Then there are definable binary operations $+$ and $\cdot$ on V such that $\langle V, +, \cdot\rangle$ is an algebraically closed field.*

Proof. Since B is solvable, V is abelian. Since B is m-connected and V is not central, V is infinite.

Every element of T acts by conjugation on V and defines an automorphism of V. In what follows we shall denote the group operation in V by $+$ and the action by conjugation of $t \in T$ on $v \in V$ by tv.

Thus, for any $t \in T$, $v_1, v_2 \in V$, we have

$$t(v_1 + v_2) = tv_1 + tv_2, \qquad t0 = 0.$$

Therefore, we interpret elements of T as additive operators on V. Let $-t$ denote the operator defined by $(-t)v = -tv$ and

$$T' = T \cup \{-t\colon\ t \in T\}.$$

For any $v \in V$ we also set

$$T' \cdot v = \{tv\colon\ t \in T'\}.$$

Since T is m-connected, $T' \cdot v$ is infinite provided v is not central, i.e. $v \neq 0$. Using the finiteness of Morley rank of V, it is easy to prove that for every $v_0 \in V$, $v_0 \neq 0$, the subgroup V_0 of V generated by $T' \cdot v_0$ is definable and there is a natural number N such that every $v \in V_0$ is of the form

$$v = t_1 v_0 + \dots + t_N v_0, \qquad t_1, \dots, t_N \in T'$$

(for a detailed proof see [5], Theorem 3.3). It is evident that V_0 is normal in B, so $V = V_0$ by the minimality of V.

Now we prove that if $t_1 v_0 + \ldots + t_k v_0 = 0$ for some $t_1, \ldots, t_k \in T'$, then $t_1 v + \ldots + t_k v = 0$ for all $v \in V$. For this we observe that for every $t \in T'$

$$t \cdot (t_1 v_0 + \ldots + t_k v_0) = t_1 \cdot t v_0 + \ldots + t_k \cdot t v_0.$$

Thus, if $t_1 v_0 + \ldots + t_k v_0 = 0$, then $t_1 v + \ldots + t_k v = 0$ for all $v \in T' \cdot v_0$ and, since V is generated by $T' \cdot v_0$, for all $v \in V$.

For any $t_1, \ldots, t_k \in T'$ we denote by $t_1 + \ldots + t_k$ the additive operator on V defined by $(t_1 + \ldots + t_k)v = t_1 v + \ldots + t_k v$ for all $v \in V$. Denote by F the ring of all operators of the form $t_1 + \ldots + t_k$ for any $t_1, \ldots, t_k \in T'$.

It follows from the fact proved above that if $f_1 v_0 = f_2 v_0$ for some $f_1, f_2 \in F$, $v_0 \in V$, $v_0 \neq 0$, then $f_1 v = f_2 v$ for any $v \in V$. Thus we get $f_1 = f_2$.

So there is a one-to-one correspondence between elements f of F and pairs $\langle v_0, v \rangle$, namely $v = f v_0$, where $v_0 \neq 0$, $v_0 \in V$, is fixed and v is an arbitrary element of V. Moreover, every non-zero element of F is invertible. Indeed, if $f v_1 = v_0$, v_0, $v_1 \neq 0$, then there is a $g \in F$ such that $g v_0 = v_1$. Thus $g \cdot f \cdot v_1 = v_1$, which implies $g \cdot f = 1$.

Finally, we summarize the above-proved facts:

F is an infinite field, interpretable in B by elements $v \in V$, provided $v_0 \in V$, $v_0 \neq 0$, is fixed. F is algebraically closed since F is ω-stable (see [3]).

COROLLARY. *If B is an m-connected solvable group which is not nilpotent, then an algebraically closed field is definable in B.*

Proof. There exists an infinite group B' which is definable in B, m-connected, solvable, has a trivial center, and is of minimal Morley rank among all the groups satisfying these conditions. This group satisfies the assumptions of Lemma 2.

2. Algebraic groups over algebraically closed fields. In this section, G is an algebraic group over an algebraically closed field K. Every such group is definable in K (for details see, e.g., [5], Section 2), and therefore G has an ω-stable theory of finite Morley rank.

Now we recall the following well-known fact (see [4], Theorem 13.3):

TARSKI'S THEOREM. *Every definable subset S of the set K^n of n-tuples of elements of K is a Boolean combination of subsets of the form*

$$\{\langle x_1, \ldots, x_n \rangle \in K^n \colon p(x_1, \ldots, x_n) = 0\},$$

where p is an n-variable polynomial with coefficients from K. In other words, $S = F \setminus E$, where $F, E \subseteq K^n$ are closed in Zariski topology.

COROLLARY. *Every definable subgroup H of G is an algebraic subgroup of the algebraic group G.*

The corollary follows from the fact that every open subgroup of G is algebraic.

In the sequel, let G be a simple algebraic group and let B be a *Borel* *subgroup* of G, i.e., a maximal solvable algebraic subgroup of G. The m-closure $\tilde{B}$ of B in G is a solvable algebraic group, and thus $\tilde{B} = B$.

Take a minimal normal subgroup V definable in B, which lies in the center of the unipotent part of B. Then $T = B/C(V)$ is a torus, and so T is abelian.

Thus B satisfies all the assumptions of Lemma 2. Consequently, as was shown in the proof of Lemma 2, T is embeddable in the field F of automorphisms of the unipotent subgroup V. The characteristic of F coincides with that of K because it is defined by the exponent of V. As was already noted, T is a torus, i.e., a product of a finite number of copies of the multiplicative group K^* of the field K. On the other hand, T is a subgroup of the multiplicative group F^* of F. Comparing the fusion parts of T, F^*, and K^* we infer that T is a one-dimensional torus. Since any torus T acting on an abelian unipotent group has a one-dimensional invariant subgroup on which T acts transitively, by the minimality of V we get the following special version of Lemma 2.

PROPOSITION 1. *Assume that V is a one-dimensional unipotent group. Let kv be the action of $k \in K \cong T$ on $v \in V$. Then V is a one-dimensional vector space over K with respect to the multiplication kv and the group operation on V is the addition. Fixing an arbitrary non-zero element $v_0 \in V$ we get a birational isomorphism $j\colon K \to \langle V, +, \cdot \rangle$ such that $j(k) = kv_0$ for any $k \in K$.*

Remark. It is worth noting that V is just the root subgroup U_d corresponding to the maximal root d of the root system R of G.

As a consequence of Proposition 1 we get the following statement:

PROPOSITION 2. *Let G_1 be a linear (not necessarily algebraic) group over an algebraically closed field K_1 such that there is an abstract isomorphism $s\colon G \to G_1$. Then the subgroups $B_1 = s(B)$ and $V_1 = s(V)$ of G_1 satisfy the assertions of Proposition 1 and there is a subfield K_1' of K_1 such that*

$$s\colon \langle V, +, \cdot \rangle \to \langle V_1, +, \cdot \rangle, \quad j_1\colon K_1' \to \langle V_1, +, \cdot \rangle, \quad j_1^{-1} \circ s \circ j\colon K \to K_1' \subseteq K_1$$

are isomorphisms of the fields.

Proof. Since B_1 is a solvable linear group, by the Kolchin–Mal'cev theorem it contains a triangularizable normal subgroup B_1° of finite index. The intersection $V_1 \cap B_1^\circ$ is non-trivial since B_1° is of finite index in B_1. Thus $V_1 \subseteq B_1^\circ$ by minimality. It is easy to see that V_1 lies in the commutant of B_1°, which is a unipotent subgroup. Moreover, V_1 lies in the center of the unipotent part of B_1 since every non-trivial normal subgroup of a nilpotent group intersects non-trivially with center.

Now, considerations analogous to those from the proof of Proposition 1 and the fact that $T_1 = B_1/C(V_1)$ acts transitively on V_1 show that V_1 is a one-dimensional algebraic unipotent group and T_1 is an algebraic torus over a field $K_1' \subseteq K_1$, which is birationally isomorphic by j_1 to $\langle V_1, +, \cdot \rangle$.

Given a field embedding $u\colon K \to K_1$, we denote by u^* the group embedding

$$u^*\colon \mathrm{GL}(n, K) \to \mathrm{GL}(n, K_1)$$

induced by u.

LEMMA 3. *Let u be a field isomorphism and j a birational isomorphism of the additive group of K onto a unipotent group V. Then there exists a birational isomorphism j_0 such that the following diagram is commutative*:

$$\begin{array}{ccc} K & \xrightarrow{\;j\;} & V \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u^*} \\ K_1 \supseteq K_1' & \xrightarrow[j_0]{} & V_1 \end{array}$$

Proof. Every one-dimensional unipotent group can be uniquely, up to birational isomorphism, interpreted as a one-dimensional vector space, and every birational isomorphism $j\colon K \to V$ is of the form $j(k) = kv_0$ for some non-zero $v_0 \in V$ and every $k \in K$. Thus

$$u^*(kv_0) = \big(u(k)\big)v_1,$$

where $v_1 = u^*(v_0)$. Therefore, $j_0\big(u(k)\big) = \big(u(k)\big)v_1$, and so j_0 is birational.

LEMMA 4. *Under the assumptions of Proposition 2 define $u = j_1^{-1} \circ s \circ j$, $G_2 = u^*(G)$, and $V_2 = u^*(V)$. Then there is a group isomorphism $r\colon G_2 \to G_1$ such that $s = r \circ u^*$ and r is birational on V_2.*

Proof. By the definition we have $j \circ u = s \circ j$, and by Lemma 3 we obtain $j_0 \circ u = u^* \circ j$ for some birational isomorphism $j_0\colon K_1' \to V_2$. Thus we have $j_1 \circ j_0^{-1} \circ u^* \circ j = s \circ j$. Putting $r = s(u^*)^{-1}$, we get $r|V_2 = j_1 \circ j_0^{-1}$, and thus r is a birational isomorphism on V_2.

Without loss of generality we can assume that K_1 is algebraically closed.

The groups G_1 and G_2 are subgroup of $\mathrm{GL}(n, K_1)$, and G_2 is an algebraic subgroup of $\mathrm{GL}(n, K_1')$. Denote by $\bar{G}_1$ and $\bar{G}_2$, respectively, closures of these groups in $\mathrm{GL}(n, K_1)$ in Zariski topology. Obviously, $\bar{G}_2$ is simple since G_2 is algebraic and simple.

LEMMA 5. *The mapping $r\colon G_2 \to G_1$ of Lemma 4 can be extended to a mapping $\bar{r}\colon \bar{G}_2 \to \bar{G}_1$ which is definable in K_1 (in the field-language) and is a group isomorphism. G_1 is a simple algebraic group over K_1.*

Proof. The least normal subgroup of G_2 containing V_2 coincides with G_2. Thus every element of G_2 is of the form $v_1^{g_1} \ldots v_k^{g_k}$ for some $v_1, \ldots, v_k \in V_2$ and $g_1, \ldots, g_k \in G_2$. Using the finiteness of Morley rank of G_2, we get a natural number N and some fixed $g_1, \ldots, g_N \in G_2$ such that every element of G_2 is of the form $v_1^{g_1} \ldots v_N^{g_N}$ for some $v_1, \ldots, v_N \in V_2$.

Define $\bar{r}$ on $\bar{V}_2$ as a birational isomorphism extending r. Put $h_i =$

$r(g_i)$, $i \in \{1, \ldots, N\}$, and consider the following binary relation between $x \in \bar{G}_2 \subseteq K_1^n$ and $y \in \bar{G}_1 \subseteq K_1^n$:

$$R(x, y) \equiv (\forall v_1, \ldots, v_N \in \bar{V}_2)\big(x = v_1^{g_1} \ldots v_N^{g_N} \leftrightarrow y = \bar{r}(v_1)^{h_1} \cdot \ldots \cdot \bar{r}(v_N)^{h_N}\big).$$

It is easy to see that $R(x, y)$ is equivalent to $r(x) = y$ for $x \in G_2$. Since $K_1' \subseteq K_1$ is an elementary extension, R defines the graph of a group isomorphism of $\bar{G}_2$ onto some subgroup of GL (n, K_1) containing G_1 and $\bar{V}_1$. Since $\bar{r}|\bar{V}_2$ is a mapping definable in K_1 and $\bar{V}_2$ is a definable subset of K_1^n, R is definable in K_1.

The last fact implies, in particular, that $\bar{r}(\bar{G}_2)$ is definable in K_1. Thus, by the Corollary to Tarski's theorem, $\bar{r}(\bar{G}_2)$ is an algebraic group over K_1, and hence $\bar{r}(\bar{G}_2) = \bar{G}_1$.

For char $(K_1) = p \neq 0$, denote by Fr_k the Frobenius automorphism Fr_k: $y \to y^{p^k}$. Set $\mathrm{Fr}_k = \mathrm{id}$ if char $(K_1) = 0$.

PROPOSITION 3. *Let H be a definable subset of K_1^n and t a definable mapping $H \to K_1$, where K_1 is an algebraically closed field. Then there are an open subset $\dot{H}$ of H, a natural number k, and a rational function $\hat{t}$: $\dot{H} \to K_1$ such that, for every $x \in \dot{H}$, $y \in K_1$,*

$$t(x) = y \quad \textit{iff} \quad \hat{t}(x) = \mathrm{Fr}_k\, y.$$

Proof. By Tarski's theorem we have

$$t(x) = y \equiv P_0(x, y) \vee \ldots \vee P_m(x, y),$$

where every $P_i(x, y)$, $i \in \{0, \ldots, m\}$, is a conjunction of polynomial equations and inequalities. Moreover,

$$H = H_0 \cup \ldots \cup H_m, \qquad \text{where } H_i = \{x \in H\colon (\exists y)\, P_i(x, y)\},\ i \in \{0, \ldots, m\}.$$

Again by Tarski's theorem, H_i are of the form $F_i \backslash E_i$, where F_i and E_i are closed in H. Hence for some $i \in \{0, \ldots, m\}$, say $i = 0$, H_i is an open subset of H. We also assume without loss of generality that the closure of H_0 is irreducible.

It is obvious that $t(x) = y \equiv P_0(x, y)$ for $x \in H_0$. Let

$$P_0(x, y) = \Big(\bigwedge_{1 \leqslant i \leqslant p} f_i(x, y) = 0 \wedge \bigwedge_{1 \leqslant j \leqslant q} g_j(x, y) \neq 0\Big).$$

Note that $p > 0$, for otherwise $P_0(x, y)$ is not a graph of a function.

Now, consider f_i $(1 \leqslant i \leqslant p)$ and g_j $(1 \leqslant j \leqslant q)$ as polynomials with one variable y and with rational functions on H_0 as coefficients. Since a ring of one-variable polynomials is a principal ideal domain, we obtain

$$P_0(x, y) \equiv \Big(f(x, y) = 0 \wedge \bigwedge_{1 \leqslant j \leqslant q} g_j(x, y) \neq 0\Big)$$

for some polynomial $f(x, y)$ of the ring.

We can assume that $f(x, y)$ is irreducible over the field of rational functions on H_0, for otherwise we could decompose $P_0(x, y)$ into a disjunction of expressions of the same kind.

Since $f(x, y)$ is irreducible, $g_j(x, y)$ $(1 \leqslant j \leqslant q)$ and $f(x, y)$ have no common solution in K_1 for every x belonging to an open subset $\dot{H}$ of H_0. Thus for $x \in \dot{H}$ we get $P_0(x, y) \equiv (f(x, y) = 0)$ and $f(x, y)$ has a unique solution in K_1 for every $x \in \dot{H}$. Consequently,

$$f(x, y) = a(x)(\mathrm{Fr}_k(y) - \hat{t}(x))$$

for some rational functions $a(x)$, $\hat{t}(x)$ and a natural number k.

LEMMA 6. *The isomorphism* $\bar{r}$: $\bar{G}_2 \to \bar{G}_1$ *of Lemma* 5 *is birational.*

Proof. The isomorphism $\bar{r}$ is an n-tuple $\langle r_1, \ldots, r_n \rangle$, where r_i: $K_1^n \to K_1$ are definable functions for all $i \in \{1, \ldots, n\}$. It follows from Proposition 3 that for every x from an open subset of $\bar{G}_2$ we have $r_i(x) = \mathrm{Fr}_{k_i}(\hat{t}_i(x))$, where $\hat{t}_i(x)$ is a rational function and k_i an integer, $i \in \{1, \ldots, n\}$. We assume that if $\mathrm{char}(K_1) = 0$, then $k_i = 0$, and if $\mathrm{char}(K_1) \neq 0$, then k_i is maximal. Thus $\hat{t}_i(x)$ is separable, i.e., $d\hat{t}_i(x) \not\equiv 0$, where $d\hat{t}_i$ is the differential of $\hat{t}_i$.

Let k be the minimal natural number such that $k + k_i \geqslant 0$ for all $i \in \{1, \ldots, n\}$. Then $\mathrm{Fr}_k \circ \bar{r}$ is rational on an open subset of G_2, and thus on G_2. Note that $d(\mathrm{Fr}_k \circ \bar{r}) \neq 0$ since $d(\mathrm{Fr}_k \circ r_i) \neq 0$ for some $i \in \{1, \ldots, n\}$. We shall prove that $k = 0$, i.e., $\mathrm{Fr}_k = \mathrm{id}$.

Suppose that $k > 0$. Then $d(\mathrm{Fr}_k \circ \bar{r})$ is a non-zero homomorphism of Lie algebras $L(\bar{G}_2) \to L(\bar{G}_1)$. But $d(\mathrm{Fr}_k \circ \bar{r})$ is a zero homomorphism on $L(\bar{V}_2)$ since $\bar{r}$ is rational on $\bar{V}_2$ and $k > 0$. Thus, the kernel of $d(\mathrm{Fr}_k \circ \bar{r})$ is non-trivial. This is a contradiction since $L(\bar{G}_2)$ is a simple Lie algebra.

Thus we have proved that $k_i \geqslant 0$ for all $i \in \{1, \ldots, n\}$, and so $\bar{r}$ is rational. By symmetry, $\bar{r}^{-1}$ is also rational, which completes the proof.

THEOREM (Borel and Tits [1]). *Let s be an isomorphism of a simple algebraic group G over an algebraically closed field K onto a linear group* G_1 *over an algebraically closed field* K_1. *Then there exist*:

an embedding of fields u: $K \to K_1$;

algebraic groups $\bar{G}_1$ *and* $\bar{G}_2$ *over* K_1, $G_1 \subseteq \bar{G}_1$;

a group embedding u^*: $G \to \bar{G}_2$ *induced by u and a rational isomorphism* r: $\bar{G}_2 \xrightarrow{\text{onto}} \bar{G}_1$ *such that* $s = r \circ u^*$.

This theorem follows immediately from Proposition 2 and Lemmas 4–6.

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