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CONNECTION BETWEEN SET THEORY AND THE FIXED POINT PROPERTY

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1. Introduction. If $f\colon X \to X$ is a transformation of a set X into itself, then every point p of X such that $p = f(p)$ is called a *fixed point* of f. There are many directions of investigation of the fixed points, mainly the existence of them, under various conditions on $f\colon X \to X$. One of these directions is given by fixed point theorems for transformations of an ordered set into itself which satisfy some order-theoretical conditions (see, e.g., [11] for a celebrated study originated by [6]; see also [10]). Another direction consists in investigation of the (topological) fixed point property, i.e., the property of a topological space X that every continuous transformation $f\colon X \to X$ has a fixed point (for an expository article, see, e.g., [1]). The fixed point property for continua (i.e., connected and compact topological spaces) of the lowest dimension constitutes a part of this topological investigation.

In [8] I made a suggestion that for this study a primary domain is given by basic fixed point theorems in ordered sets.

The aim of the present paper is to verify this hypothesis* by stating an order-theoretical fixed point theorem (Theorem 1 in Section 2) and deriving from Theorem 1 a fixed point theorem for some Hausdorff continua (Theorem 2 in Section 3) without any use of the axiom of choice (contrary to the usual proofs in the literature related to this subject). The final theorems (Theorems 3–5 in Section 4) give a discussion of the method introduced here, by Theorem 1, with respect to basic theorems of the elementary set theory. Theorems 3 and 4 do not also depend on the axiom of choice.

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2. A fixed point theorem in partially ordered sets. A partially ordered set X, with an ordering $\leqslant$, will be called *inductively ordered* or, shortly, *inductive*

* The main result of this work was presented at Colloquium on Topology, Eger, Hungary, August 12, 1983.

if for every totally ordered subset of X there exists a least upper bound of this subset in X. Then also the empty set $\emptyset$ has a least upper bound, $\sup \emptyset$, which is a least element in X (so that every inductive set is in particular non-empty). X will be called *acyclically ordered* if for every $p, r \in X$ with $p \leqslant r$ the segment $[p, r] = \{q \in X\colon\ p \leqslant q \leqslant r\}$ is totally ordered.

We shall consider the transformations $f\colon X \to X$ which satisfy the following two conditions:

(I) $p < f(p)$ implies the existence of $q \in (p, f(p)]$ with $q \leqslant f(q)$.

(II) $q \leqslant f(q)$ for all $q \in Y$ implies $\sup Y \leqslant f(\sup Y)$ if there is $\sup Y$.

THEOREM 1. *If X is a set inductively and acyclically ordered by a partial order $\leqslant$ and a transformation $f\colon X \to X$ satisfies* (I) *and* (II), *then there exists a fixed point of f.*

Proof. Since X is inductive by assumption, the set

$$P_f = \{q \in X\colon\ q \leqslant f(q)\} \tag{2.1}$$

is also inductively ordered by the order $\leqslant$, in view of (II). It follows by the assumed acyclicity of X that for every $p \leqslant f(p)$ there exists in P_f supremum of the set $[p, f(p)] \cap P_f$. Therefore the function

$$\varphi(p) = \sup\big([p, f(p)] \cap P_f\big), \quad \text{where } p \in P_f, \tag{2.2}$$

transforms P_f into itself. Since $p \leqslant \varphi(p)$ for all $p \in P_f$ by (2.1) and (2.2), it follows (by [3], pp. 434–435) that there exists a fixed point of φ.

Now it suffices to verify that for every $p \in P_f$

$$p = \varphi(p) \text{ implies } p = f(p). \tag{2.3}$$

Suppose, on the contrary, that $p \neq f(p)$, i.e. (since $p \in P_f$) that $p < f(p)$. Then by (I) and (2.1) there exists $q \in (p, f(p)] \cap P_f$. Hence $p < q$ and, by (2.2), $q \leqslant \varphi(p)$. It follows that $p < \varphi(p)$, contrary to the predecessor of the implication (2.3).

3. An application to the topological fixed point property. By a *continuum* we mean here an arbitrary non-empty connected and compact Hausdorff space. Any continuum having exactly two points which do not disconnect it is said to be an *arc* (including continua consisting of one point only). A continuum X is said to be *arcwise connected* if for any points $p, q \in X$ there exists an arc joining these points in X; X is said to be *one-arcwise connected* if the arc is unique in X, and the arc will be denoted by pq.

In this section, X will be an arbitrary one-arcwise connected continuum such that for every monotone family of arcs $ap_\tau \subseteq X$, $\tau \in T$, there exists $b \in X$ such that

$$\overline{\bigcup_{\tau \in T} ap_\tau} = ab$$

(such a continuum X is sometimes called a *one-arcwise connected nested continuum*).

The continua considered in [2] belong to the class considered here, by Lemma 1 in [2] (see also [7], Remark 2, p. 109, for a simple proof in a more general setting). The following standard example proves that the class of continua considered here is essentially larger, also in the metric case:

EXAMPLE 1. A cone over an arbitrary hereditarily indecomposable plane continuum is a one-arcwise connected nested continuum which is not hereditarily unicoherent.

For an arbitrary point $a \in X$, a partial order $\leqslant_a$ is determined in the nested continuum X: for any $p, q \in X$

$$p \leqslant_a q \text{ if and only if } ap \subseteq aq.$$

In the order $\leqslant_a$, the point a is of course a smallest element of X, and a set $\{p_\tau \in X: \tau \in T\}$ is totally ordered if and only if the family of arcs $\{ap_\tau \subseteq X: \tau \in T\}$ is monotone.

LEMMA. *For an arbitrary point $a \in X$, the nested continuum X is inductively ordered by the order $\leqslant_a$.*

Proof. Let $\{ap_\tau \subseteq X: \tau \in T\}$ be a monotone family of arcs so that there exists $b \in X$ with $\overline{\bigcup_{\tau \in T} ap_\tau} = ab$. Then $ap_\tau \subseteq ab$ for all $\tau \in T$, and therefore b is an upper bound of the totally ordered set of points $p_\tau \in X$. To prove that b is the least upper bound, let $ap_\tau \subseteq ac$ for all $\tau \in T$. Then

$$\overline{\bigcup_{\tau \in T} ap_\tau} \subseteq ac,$$

the arc ac being closed in X as a compact subset of X, i.e., $ab \subseteq ac$.

Remark. It is stated in the proof of the Lemma that the equality $\overline{\bigcup_{\tau \in T} ap_\tau} = ab$ implies that b is the supremum in X (with the order $\leqslant_a$) of the set $\{p_\tau \in X: \tau \in T\}$. The converse also holds, i.e., b is the supremum of a set $\{p_\tau \in X: \tau \in T\}$ if and only if the equality $\overline{\bigcup_{\tau \in T} ap_\tau} = ab$ holds.

Indeed, if b is this supremum, then $ap_\tau \subseteq ab$ for all $\tau \in T$. Hence, ab being closed in X, $\overline{\bigcup_{\tau \in T} ap_\tau} \subseteq ab$. But the closure $\overline{\bigcup_{\tau \in T} ap_\tau}$ is a subarc ac of ab and $ap_\tau \subseteq ac$ for all $\tau \in T$. Thus $ac \subseteq ab$ and c is an upper bound of the set $\{p_\tau \in X: \tau \in T\}$. Since b is the least upper bound of this set, $c = b$.

Further, some geometrically intuitive facts will also be needed, concerning another relation between arcs lying in X (which is investigated in [7] in some more general setting).

Namely, for every two arcs pq, $pr \subseteq X$, with the same initial point, the association $pq \prec pr$ is defined by the following formula: $pq \cap pr \neq \{p\}$, i.e., in view of the one-arcwise connectedness of X, by the statement that $pq \cap pr$ is an arc non-degenerate to the point p, thus a common initial arc of pq and pr.

The association $\prec$ is an equivalence relation in the family of all non-degenerate arcs with the same initial point (see [7], Proposition 2, p. 108), and of course $pq \subseteq pr$ implies $pq \prec pr$ for $p \neq q$.

Since for every arcwise connected continuum $K \subseteq X$

$$q, r \in K \text{ implies } qr \subseteq K, \tag{3.1}$$

we have

$$p \notin K \text{ and } q, r \in K \text{ imply } pq \prec pr \tag{3.2}$$

(the arcwise connectedness can be omitted in (3.2), but it will be needed in the sequel, in connection with (3.1)). Moreover,

$$ap \subseteq aq \text{ and } pq \prec pr \text{ imply } ap \subseteq ar \tag{3.3}$$

(see [7], Proposition 6, p. 108) and it is worth at once noting that, in view of (3.1), the inclusion $ap \subseteq aq$ means that $p \in aq$, which is further equivalent to the equality $ap \cup pq = aq$, so that, in particular, $ap \subseteq aq$ implies $pq \subseteq aq$.

Now the above facts can be applied to the following

THEOREM 2. *Every one-arcwise connected nested continuum X with the partial order $\leqslant_a$ is an inductively and acyclically ordered set and every transformation $f\colon X \to X$ such that*

(III) *$f(pq)$ is an arcwise connected continuum for each pq,*

(IV) *for each $p \neq f(p)$ there is $\{p\} \neq pq \subseteq pf(p)$ with $pq \cap f(pq) = \emptyset$, satisfies conditions* (I) *and* (II) *in the order $\leqslant_a$ (thus there exists a fixed point of f by virtue of Theorem* 1).

Proof. By the Lemma, X is inductively ordered in the order $\leqslant_a$. The acyclicity of X follows directly from the one-arcwise connectedness of X, because for every $p \leqslant_a q$ the set $[p, q]$ is equal to the arc pq, and the order $\leqslant_a$ in pq is the natural order of the arc pq (which can be defined in pq as $\leqslant_p$).

Now conditions (I) and (II) will be verified for the transformation $f\colon X \to X$ satisfying (III) and (IV) by assumption.

(I) Suppose that

$$ap \not\subseteq af(p) \tag{3.4}$$

so that $p \neq f(p)$, and hence by (IV) there is $q \in X$ such that

$$\{p\} \neq pq \subseteq pf(p), \tag{3.5}$$

$$pq \cap f(pq) = \emptyset. \tag{3.6}$$

It follows, by (3.4) and (3.5), that $ap \not\subseteq aq \subseteq af(p)$, and it remains to prove that $aq \subseteq af(q)$.

From (3.6) it follows that $p, q \notin f(pq)$. Since $f(pq)$ is an arcwise continuum by (III), we have

$$qf(p) \prec qf(q) \tag{3.7}$$

and $pf(p) \prec pf(q)$ in view of (3.2). Since $pq \prec pf(p)$ (by (3.5)), it follows that $pq \prec pf(q)$ by the transitivity of the association $\prec$. Hence, by (3.3) and (3.4), $ap \subseteq af(q)$; consequently, $pf(q) \subseteq af(q)$. Simultaneously, by (3.3), (3.5) and (3.7), $pq \subseteq pf(q)$. Therefore $pq \subseteq af(q)$, thus $q \in af(q)$, i.e., $aq \subseteq af(q)$.

(II) Let $\{ap_\tau \in X\colon \tau \in T\}$ be an arbitrary monotone family with

$$ap_\tau \subseteq af(p_\tau) \quad \text{for all } \tau \in T \tag{3.8}$$

and let $b \in X$ be the supremum in X (with the order $\leqslant_a$) of the set $\{p_\tau \in X\colon \tau \in T\}$, i.e., by the Remark, let

$$ab = \overline{\bigcup_{\tau \in T} ap_\tau}. \tag{3.9}$$

Suppose, on the contrary, that the inclusion $ab \subseteq af(b)$ does not hold, so that $p_\tau \neq b$ for all $\tau \in T$ in view of (3.8). Then $b \notin af(b)$, which implies $ba \prec bf(b)$ by (3.2). Since $b \neq f(b)$ and $\bigcap_{\tau \in T} bp_\tau = \{b\}$ by (3.9), being $bp_\tau \subseteq bf(b)$ for bp_τ sufficiently small, by (IV) there exists $\tau \in T$ such that $bp_\tau \cap f(bp_\tau) = \emptyset$. Therefore we infer that even

$$ab \cap f(p_\tau b) = \emptyset. \tag{3.10}$$

Indeed, in the opposite case, since $p_\tau \notin f(p_\tau b)$, we would have $ap_\tau \cap f(p_\tau b) \neq \emptyset$ in view of (3.9). Thus taking in the arc ap_τ the first point p belonging to $f(p_\tau b)$, the image being a continuum by (III), we would have $p_\tau \notin ap$. Consequently, $p_\tau \notin ap \cup f(p_\tau b)$. But $af(p_\tau) \subseteq ap \cup f(p_\tau b)$ in view of (3.1) and (III). Hence $p_\tau \notin af(p_\tau)$, contrary to (3.8), which proves (3.10).

Consider the continua ab and $f(p_\tau b)$ disjoint by (3.10). By the one-arcwise connectedness of X, there exists in X a unique arc joining them so that $ab \cap pq = \{p\}$ and $pq \cap f(p_\tau b) = \{q\}$, and any arc joining an arbitrary point of ab with an arbitrary point of $f(p_\tau b)$ contains pq. In particular, $p \in af(b)$, and since $b \notin af(b)$ by the assumption on the contrary, we have $p \neq b$. But in view of (3.9), $ap \cup pb = ab$. Thus taking eventually an arc $p_{\tau'} b \subseteq ab$ smaller than the arc $p_\tau b$, and appropriate $q' \in f(p_{\tau'} b)$ instead of q, it may be supposed without change of notation (in (3.10)) that $ap \cap p_\tau b = \emptyset$.

But then $pq \cap p_\tau b = \emptyset$ by the definition of the arc pq, and since $qf(p_\tau) \subseteq f(p_\tau b)$ by (3.1) and (III), we have in view of (3.9) and (3.10) also $qf(p_\tau) \cap p_\tau b = \emptyset$. Consequently,

$$\big(ap \cup pq \cup qf(p_\tau)\big) \cap p_\tau b = \emptyset.$$

Since $af(p_\tau) \subseteq ap \cup pq \cup qf(p_\tau)$ according to (3.1) (in fact, the equality can be even proved), we obtain $p_\tau b \cap af(p_\tau) = \emptyset$. Hence $p_\tau \notin af(p_\tau)$, contradicting (3.8).

Theorem 2 implies the topological fixed point property of arcwise connected hereditarily unicoherent continua [2], because every continuous transformation $f\colon X \to X$ satisfies of course (III) and (IV). These conditions are however essentially more general than the continuity as will be shown by the following

EXAMPLE 2. Let X be a plane continuum described as the union of the segment I_0 with the ends $(0, 0)$ and $(1, 0)$ and segments I_n with the ends $(0, 0)$ and $(1, 1/n)$ for $n = 1, 2, \ldots$ Define $f\colon X \to X$ as follows: let $f(p) = p$ for $p \in I_n$ and $n = 1, 2, \ldots$, and for $p \in I_0$, i.e., for $p = (t, 0)$, where $0 \leqslant t \leqslant 1$, let $f(p) = (2t, 0)$ whenever $0 \leqslant t \leqslant \frac{1}{2}$ and $f(p) = (1, 0)$ whenever $\frac{1}{2} \leqslant t \leqslant 1$. Then all the assumptions of Theorem 2 are satisfied, and the transformation $f\colon X \to X$ is not continuous (other but related classes of fixed point transformations, more general than continuous ones, are discussed in the examples of [9], pp. 125–128).

4. Connection with basic fixed point theorems in partially ordered sets. By the *basic fixed point theorems in partially ordered sets* we understand the ones which can go to make a basis for an introductory discourse of set theory. Namely, these theorems state the existence of a fixed point of a transformation f of an inductively ordered set into itself if f is:

1° increasing, i.e., such that $f(p) \leqslant f(q)$ whenever $p \leqslant q$ (the Knaster–Tarski theorem [6]; see also [11] and [4]).

2° progressive, i.e., such that $p \leqslant f(p)$ for all points p (the fixed point theorem proved by Bourbaki [3] as a refinement of an idea of Zermelo [13]).

First of all, the following strong connection between these two fixed point theorems is to be noted (for a transformation $f\colon X \to X$, a subset $P \subseteq X$ will be called *f-invariant* if $f(P) \subseteq P$).

THEOREM 3. *If a transformation $f\colon X \to X$ of an inductively ordered set X into itself is progressive or increasing, then an inductive f-invariant subset of X is determined on which f is both progressive and increasing.*

Proof. For f progressive and for an arbitrary point $a \in X$, let $P_f(a)$ denote the common part of all f-invariant subsets $P \subseteq X$ which are inductive and contain the point a (it is worth to realize that $\{a, f(a), f(f(a)), \ldots\} \subseteq P_f$). Then $P_f(a)$ is f-invariant and inductive, being even a complete lattice as a totally ordered inductive subset of X (by Theorem 1 of [3]). Moreover, it follows (by the property (P) in [3], p. 434, and the equality proved at the end of the proof of Theorem 1 of [3]) that the transformation f is increasing and progressive on $P_f(a)$.

If f is increasing, then f satisfies (II) by a standard argument (see, e.g., [4], p. 14) and also f satisfies the condition

(I′) $p \leqslant f(p)$ implies $f(p) \leqslant f(f(p))$

directly by the definition given in 1°. Condition (I′) means that the set P_f defined by (2.1) is f-invariant and, by (II), P_f is inductive. Thus, in view of (2.1), f is both progressive and increasing on P_f.

Theorem 3 implies that an exposition of the basic fixed point theorems in partially ordered sets can be made (without any use of the axiom of choice) by proving the fixed point theorem for progressive transformations and deriving from it the fixed point theorem for increasing transformations or, conversely, by stating the theorem for increasing transformations (in complete lattices only), and then considering progressive transformations. Then the basic theorems of elementary set theory, i.e. the Cantor-Bernstein theorem and the Kuratowski–Zorn lemma, follow easily (see, e.g., [3], [6], [11], p. 305; see also [10]). It is also to note that the Kuratowski–Zorn lemma is in a simple equivalence with the conjunction of the axiom of choice and the fundamental fixed point theorems.

Now the basic fixed point theorems will be compared with Theorem 1. Every increasing or progressive transformation $f\colon X \to X$ of an ordered set X into itself satisfies (I′) and (II), thus also (I) and (II). Hence Theorem 1 is a generalization of the basic fixed point theorems under the additional assumption of the acyclicity of the ordered set X.

Further implications of the argument given in Theorem 1 are also interesting, and the main one seems to be the following

PROBLEM. Extend Theorem 1 to preordered sets, i.e., ordered by a relation which is reflexive and transitive only, so that it will imply the topological fixed point theorem [7].

Let us note that a formulation of the theorem [7] in lattice theory is given in [5]. Thus the above problem requires another order-theoretical investigation of the theorem [7]. For lattices the following seems to be interesting, generalizing the classical version of the Knaster–Tarski theorem [11] by an analogous proof as that for Theorem 1.

THEOREM 4. *If X is a complete lattice and a transformation $f\colon X \to X$ satisfies* (I) *and* (II), *then there exists a fixed point of f.*

For a set X which is inductively ordered only, as in the basic theorems, the following generalization of them, and of Theorems 1 and 4, holds (by a standard application of the Kuratowski–Zorn lemma).

THEOREM 5. *If X is an inductively ordered set and a transformation $f\colon X \to X$ satisfies* (I) *and* (II), *then there exists a fixed point of f.*

Namely, a maximal element of the set P_f (cf. (2.1)) is a fixed point.

Postscript. A kind of acyclically ordered sets, under another name, was considered in a monograph *Set theory with an introduction to descriptive set theory* by K. Kuratowski and A. Mostowski – see the second, completely revised edition, 1976, p. 84.

The fixed point theorem proved by Bourbaki, the formulation and the proof as well, does not differ essentially from the one which had been given by K. Kuratowski in his paper *Une méthode d'élimination des nombres* *transfinis des raisonnements mathématiques* (Fund. Math. 3 (1922), pp. 76–108; see p. 83, Théorème III, and p. 86, Corollaires I and I′; see also p. 77 for origins of this method). A dual proof of this fixed point theorem can be derived from the paper of E. Zermelo *Beweis, dass jede Menge wohlgeordnet werden kann* (Math. Ann. 59 (1904), pp. 514–516).

Some related, set-theoretical aspects of the fixed point theory were recently described by N. Brunner in his paper *Topologische Maximalprinzipien* (to appear in Z. Math. Logik Grundlagen Math.) and by the author in the paper *Generalized notion of the supremum* (presented at the Sixth Prague Topological Symposium 1986).

REFERENCES

[1] R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly 76 (1969), pp. 119–132.
[2] K. Borsuk, *A theorem on fixed points*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 2 (1954), pp. 11–20.
[3] N. Bourbaki, *Sur le théorème de Zorn*, Arch. Math. 2 (1949–1950), pp. 434–437.
[4] J. Dugundji and A. Granas, *Fixed point theory. I*, Warszawa 1982.
[5] R. Gurevič, *A general point of view on λ-dendroids and fixed point theorems* (in Russian), Fund. Math. 100 (1978), pp. 109–118.
[6] B. Knaster, *Un théorème sur les fonctions d'ensembles*, Ann. Soc. Polon. Math. 6 (1927–1928), pp. 133–134.
[7] R. Mańka, *Association and fixed points*, Fund. Math. 91 (1976), pp. 105–121.
[8] – *Fixed point theorem for λ-dendroids*, pp. 309–311 in: *Proceedings of the International Conference on Geometric Topology 1978*, Warszawa 1980.
[9] – *Fixed point theorems for λ-dendroids*, Fund. Math. 108 (1980), pp. 119–128.
[10] R. Sikorski, *On a generalization of theorems of Banach and Cantor–Bernstein*, Colloq. Math. 1 (1948), pp. 140–144.
[11] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math. 5 (1955), pp. 285–309.
[12] L. E. Ward, Jr., *Recent development in dendritic spaces and related topics*, pp. 601–647 in: *Studies in Topology* (*Proceedings of the Conference of the University of North Carolina, Charlotte*), New York 1975.
[13] E. Zermelo, *Neuer Beweis für die Möglichkeit einer Wohlordnung*, Math. Ann. 65 (1908), pp. 107–128.

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