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ON DINI DERIVATIVES AND ON CERTAIN CLASSES OF ZAHORSKI

BY

S. N. MUKHOPADHYAY (BURDWAN, WEST BENGAL)

1. Zahorski [4] considered a hierarchy of classes $\mathcal{M}_i$ of functions $(0 \leqslant i \leqslant 5)$ and proved that the derivative f' — finite or infinite — of continuous function f belongs to the class $\mathcal{M}_2$. He also showed that if f' is finite everywhere, then $f' \in \mathcal{M}_3$, and if further f' is bounded, then $f' \in \mathcal{M}_4$. In the present paper the above-mentioned properties are studied by considering Dini derivatives. Also a sufficient condition is obtained under which a function of the class $\mathcal{M}_1$ should belong to the class $\mathcal{M}_2$.

Throughout the paper f will denote a real function defined on the real line and $\mu(E)$ will denote the Lebesgue measure of measurable set E.

For convenience of the reader we recall definitions of Zahorski [4].

Definitions. A non-empty subset E of the real line belongs to $\mathcal{M}_0$ (resp. $\mathcal{M}_1$) if and only if E is an F_σ-set and every point of E is a bilateral point of accumulation (resp. condensation) of E. A set E belongs to $\mathcal{M}_2$ if and only if E is an F_σ-set and every one-sided neighbourhood of each point of E intersects E in a set of positive measure. A set E belongs to $\mathcal{M}_3$ if and only if E is an F_σ-set and there exist a sequence of closed sets $\{F_n\}$ and a sequence of numbers $\{\eta_n\}$, $0 \leqslant \eta_n < 1$, such that $E = \bigcup_{n=1}^{\infty} F_n$ and, for each $x \in F_n$ and every $c > 0$, there is a number $\varepsilon(x, c) > 0$ satisfying the following property: for any two numbers h and h_1 such that $hh_1 > 0$, $h/h_1 < c$, $|h+h_1| < \varepsilon(x, c)$, the following relation is true:

$$\frac{\mu\big(E \cap (x+h, x+h+h_1)\big)}{|h_1|} > \eta_n.$$

A set E belongs to $\mathcal{M}_4$ if and only if $E \in \mathcal{M}_3$ with $\eta_n > 0$ for all n.

A function $f \in \mathcal{M}_i$ $(0 \leqslant i \leqslant 4)$ if and only if for any real number a each of the sets $\{x: f(x) > a\}$ and $\{x: f(x) < a)$ belongs to the class $\mathcal{M}_i$.

2. THEOREM 1. *Let f be such that*

(i) *for all* ξ, $\lim\limits_{x\to\xi-0} f(x) = f(\xi)$, $\liminf\limits_{x\to\xi+0} f(x) \leqslant f(\xi) \leqslant \limsup\limits_{x\to\xi+0} f(x)$,

(ii) $D^- f \geqslant D^+ f \geqslant D_- f$,

(iii) *D^+f is of Baire class* 1,

(iv) $-\infty < D^+f < \infty$ *holds, except possibly for an enumerable set.*

Then $D^+f \in \mathcal{M}_2$.

Proof. Let a be arbitrary and let $x_0 \in E = \{x: D^+f(x) > a\}$. Then $D^+f(x_0) > a$. If $a = \infty$, then E is void, and if $a = -\infty$, then E represents the whole real line except possibly an enumerable set; hence the conclusion follows. So we may suppose that a is finite. Assume that there is a left neighbourhood $(x_0-\delta, x_0]$ of x_0 such that $\mu\big(E\cap(x_0-\delta, x_0]\big) = 0$. Then $D^+f(x) \leqslant a$ for almost all $x \in (x_0-\delta, x_0]$. Hence, by a result of [2], the function $f(x)-ax$ would be non-increasing in $(x_0-\delta, x_0]$. Thus $D^-f(x_0) \leqslant a$, whence, by (ii), $D^+f(x_0) \leqslant a$, which is a contradiction. Hence E intersects every left neighbourhood of x_0 in a set of positive measure. Similarly, E intersects every right neighbourhood of x_0 in a set of positive measure. Since D^+f is of Baire class 1, E is an F_σ-set. Thus $E \in \mathcal{M}_2$. Similarly, the set $\{x: D^+f(x) < a\} \in \mathcal{M}_2$. This completes the proof.

COROLLARY (Zahorski). *If f is continuous and if f' exists, finite or infinite, then $f' \in \mathcal{M}_2$.*

If condition (iv) is satisfied, then the proof follows from Theorem 1. If (iv) is not satisfied, then the proof is similar except that we have to apply a result of Goldowsky and Tonelli ([3], p. 206) instead that of [2].

LEMMA. *Under the hypotheses of Theorem 1, D^+f satisfies the mean value property, i.e., for $a < b$ there is a ξ such that $a < \xi < b$ and $f(b)-f(a) = (b-a)D^+f(\xi)$.*

Proof. It suffices to suppose that $f(a) = f(b)$ and to show that there is a ξ, $a < \xi < b$, such that $D^+f(\xi) = 0$.

If $D^+f(x) \geqslant 0$ for all $x \in (a, b)$, then f is non-decreasing in $[a, b]$, hence f is constant on $[a, b]$ which proves our assertion. Similarly, if $D^+f(x) \leqslant 0$ for all $x \in (a, b)$, then also the assertion follows. So we may suppose that there are points x' and x'' in (a, b) such that $D^+f(x') > 0$ and $D^+f(x'') < 0$. Since $D^+f \in \mathcal{M}_2$, it satisfies Darboux property [4] and hence there is $\xi \in (a, b)$ such that $D^+f(\xi) = 0$.

THEOREM 2. *Under the hypotheses of Theorem 1, if there exists a point x_0 at which $f'(x_0)$ is finite, then, for every a, $a < f'(x_0)$ (resp. $a > f'(x_0)$) the set $\{x: D^+f(x) > a)$ (resp. $\{x: D^+f(x) < a\}$) satisfies the following property: for every $c > 0$ there exists an $\varepsilon > 0$ such that for all h and h_1 with $hh_1 > 0$, $h/h_1 < c$, $|h+h_1| < \varepsilon$, we have*

$$\mu(I\cap\{x: D^+f(x) > a\}) > 0 \quad (\textit{resp. } \mu(I\cap\{x: D^+f(x) < a\}) > 0),$$

where I is the open interval with the end points x_0+h and x_0+h+h_1.

Proof. Since $f'(x_0)$ exists, taking $\lambda = f'(x_0)$ we can write

$$f(x_0+h) = f(x_0)+\lambda h+\varepsilon_1 h, \tag{1}$$

$$f(x_0+h+h_1) = f(x_0)+\lambda(h+h_1)+\varepsilon_2(h+h_1), \tag{2}$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $h \to 0$ and $h+h_1 \to 0$, respectively. If $a = -\infty$, then, for any interval I,

$$\mu(I \cap \{x\colon D^+f(x) > a\}) = \mu(I)$$

and the theorem is proved. So suppose $-\infty < a < \lambda$. Choose β, $\lambda < \beta < \infty$. Let $c > 0$ be arbitrary. Suppose $hh_1 > 0$, $h/h_1 < c$. Then $h \to 0$ as $h+h_1 \to 0$ and hence there is $\varepsilon > 0$ such that

$$c|\varepsilon_2 - \varepsilon_1| + |\varepsilon_2| < \min\left(\frac{\lambda - a}{2}, \frac{\beta - \lambda}{2}\right), \tag{3}$$

whenever $|h+h_1| < \varepsilon$.

So if $hh_1 > 0$, $h/h_1 < c$ and $|h+h_1| < \varepsilon$, we infer from (1), (2) and (3) that

$$\left|\frac{f(x_0+h+h_1)-f(x_0+h)}{h_1} - \lambda\right|$$
$$\leqslant \frac{h}{h_1}|\varepsilon_2 - \varepsilon_1| + |\varepsilon_2| < c|\varepsilon_2 - \varepsilon_1| + |\varepsilon_2| < \min\left(\frac{\lambda - a}{2}, \frac{\beta - \lambda}{2}\right),$$

i.e.

$$a < \frac{a+\lambda}{2} < \frac{f(x_0+h+h_1)-f(x_0+h)}{h_1} < \frac{\beta+\lambda}{2} < \beta. \tag{4}$$

Let the open interval with end points x_0+h and x_0+h+h_1 be denoted by I. Then from the lemma we see that there is an $\eta \in I$ such that

$$D^+f(\eta) = \frac{f(x_0+h+h_1)-f(x_0+h)}{h_1}$$

and hence, from (4), that $a < D^+f(\eta)$, i.e., $\eta \in I \cap \{x\colon D^+f(x) > a\}$. Since $D^+f \in \mathscr{M}_2$, we conclude

$$\mu(I \cap \{x\colon D^+f(x) > a\}) > 0.$$

The proof for the set $\{x\colon D^+f(x) < a\}$ can be completed similarly by interchanging a and β.

COROLLARY (Zahorski). *If f has everywhere a finite derivative f', then $f' \in \mathscr{M}_3$.*

THEOREM 3. *Let f be such that*

(i) *f is continuous,*
(ii) *D^+f is of Baire class 1,*
(iii) $-K \leqslant D^+f \leqslant K$, $0 \leqslant K < \infty$,
(iv) $D^-f \geqslant D^+f \geqslant D_-f$.

If a point x_0 is such that $f'(x_0)$ exists, then for every a, $a < f'(x_0)$ (resp. $a > f'(x_0)$), the set $\{x\colon D^+f(x) > a\}$ (resp. $\{x\colon D^+f(x) < a\}$) satisfies the *hollowing property: for every $c>0$ there exists $\varepsilon>0$ such that for all h and f_1 with $hh_1>0$, $h/h_1<c$ and $|h+h_1|<\varepsilon$, we have*

$$\frac{\mu(I\cap\{x: D^+f(x)>a\})}{|h_1|}>\min\left(\frac{1}{2},\frac{f'(x_0)-a}{2(K+|a|)}\right)$$

$$\left(\textit{resp.}\ \frac{\mu(I\cap\{x: D^+f(x)<a\})}{|h_1|}>\min\left(\frac{1}{2},\frac{a-f'(x_0)}{2(K+|a|)}\right)\right),$$

where I is the open interval with end points x_0+h and x_0+h+h_1.

Proof. Let $a<f'(x_0)$ and let $E=\{x: D^+f(x)>a\}$. If $-\infty\leqslant a<-K$, then for any interval I there is $I\cap E=I$ and hence $\mu(I\cap E)/\mu(I)>1/2$. So Theorem 3 is proved if $a=-\infty$ or $K=0$. Suppose that a is finite and $K>0$. We may assume that $a=0$, for if $a\neq 0$ we are to consider the function $f(x)-ax$ instead of $f(x)$. Put $\lambda=f'(x_0)$. Then $\lambda>0$. Let $c>0$ be arbitrary. Then, as in Theorem 2, there exists an $\varepsilon>0$ such that for all h and h_1 with $hh_1>0$, $h/h_1<c$ and $|h+h_1|<\varepsilon$, we have

$$0<\frac{\lambda}{2}<\frac{f(x_0+h+h_1)-f(x_0+h)}{h_1}. \tag{1}$$

Let I be the open interval with end points x_0+h and x_0+h+h_1. Since $|D^+f(x)|\leqslant K$ for all x, $f'(x)$ exists almost everywhere in I. Let I_0 be the subset of I consisting of points where $f'(x)$ exists. Then $\mu(I)=\mu(I_0)$. Putting

$$\Phi_n(x)=n\left\{f\left(x+\frac{1}{n}\right)-f(x)\right\},$$

we see that the sequence $\{\Phi_n(x)\}$ converges everywhere in I_0 to the function $f'(x)$. Since, by the lemma, D^+f satisfies the mean value property, we have

$$\Phi_n(x)=n\left\{f\left(x+\frac{1}{n}\right)-f(x)\right\}=D^+f\left(x+\frac{\Theta}{n}\right),\quad 0<\Theta<1,$$

and hence

$$|\Phi_n(x)|=\left|D^+f\left(x+\frac{\Theta}{n}\right)\right|\leqslant K$$

for all x and all n. So, by the theorem of Lebesgue,

$$\lim_{n\to\infty}\int_{I_0}\Phi_n(x)\,dx=\int_{I_0}f'(x)\,dx. \tag{2}$$

Since f is continuous, we infer by an easy calculation that

$$\lim_{n\to\infty}\int\Phi_n(x)\,dx=\begin{cases}f(x_0+h+h_1)-f(x_0+h) & \text{for } h_1>0,\\ f(x_0+h)-f(x_0+h+h_1) & \text{for } h_1<0.\end{cases} \tag{3}$$

Hence, from (2) and (3),

$$(4)\quad \frac{|h_1|}{h_1}\{f(x_0+h+h_1)-f(x_0+h)\} = \int_{I_0} f'(x)\,dx$$

$$= \int_I D^+f(x)\,dx \leqslant \int_{I\cap E} D^+f(x)\,dx \leqslant K\mu(I\cap E),$$

and, from (1) and (4),

$$0 < \frac{\lambda}{2K} < \frac{\mu(I\cap E)}{|h_1|},$$

thus proving Theorem 3.

If $a > f'(x_0)$, the proof is similar.

COROLLARY (Zahorski). *If f has a bounded derivative f', then $f' \in \mathscr{M}_4$.*

Proof. We may suppose a to be finite. Let $K > \frac{1}{2} - |a|$ be such that $|f'(x)| \leqslant K$ for all x. Now

$$\{x\colon f'(x) > a\} = \bigcup_{n=1}^{\infty}\left\{x\colon f'(x) > a+\frac{1}{n}\right\}.$$

Since f' is of Baire class 1, the set $\{x\colon f'(x) > a+1/n\}$ is an F_σ-set for each n. Hence

$$\left\{x\colon f'(x) > a+\frac{1}{n}\right\} = \bigcup_{m=1}^{\infty} F_{mn},$$

where F_{mn} is closed for each m. Consequently,

$$\{x\colon f'(x) > a\} = \bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} F_{mn},$$

where F_{mn} is closed for each m and n. Now considering the sequence $\{\eta_{mn}\}$, where $\eta_{mn} = 1/\{2n(K+|a|)\}$, we see that if $x_0 \in F_{mn}$, then $f'(x_0) > a+$ $+1/n$, and so from Theorem 3 we infer that for every $c > 0$ there is an $\varepsilon > 0$ such that for all h and h_1 with $hh_1 > 0$, $h/h_1 < c$ and $|h+h_1| < \varepsilon$, we have

$$(1)\quad \frac{\mu(I\cap\{x\colon f'(x) > a\})}{|h_1|} > \frac{f'(x_0)-a}{2(K+|a|)} > \frac{1}{2n(K+|a|)},$$

where I is the interval with end points x_0+h and x_0+h+h_1.

Writing double sequences $\{F_{mn}\}$ and $\{\eta_{mn}\}$ in terms of simple sequences, we obtain a sequence of closed sets $\{F_n\}$ and a sequence of numbers $\{\eta_n\}$, $0 < \eta_n < 1$, and hence from (1) we conclude that $\{x\colon f'(x) > a\} \in \mathscr{M}_4$. Similar argument is applicable for the set $\{x\colon f'(x) < a\}$.

3. Zahorski proved that the class $\mathscr{M}_0$ (or $\mathscr{M}_1$) is identical with the class of all Darboux Baire 1 functions [4]. Croft [1] has constructed a lower semicontinuous function which has Darboux property and which is zero almost everywhere but not identically zero. This shows that there are functions of the class $\mathscr{M}_0$ (or $\mathscr{M}_1$) which do not belong to $\mathscr{M}_2$. So it is natural to ask how much stronger a condition should be imposed on a function f of the class $\mathscr{M}_0$ so that f would belong to the class $\mathscr{M}_2$.

We shall prove the following.

THEOREM 4. *If $f \in \mathscr{M}_0$ and if for an arbitrary perfect set P of measure zero the set $f(P)$ does not contain an interval, then $f \in \mathscr{M}_2$.*

Proof. Choose a arbitrary and fix it. Let $E = \{x\colon f(x) > a\}$ and let $x_0 \in E$. Assume that there is a right neighbourhood $[x_0, x_0 + \delta)$ of x_0 such that $\mu(E \cap [x_0, x_0 + \delta)) = 0$. The set $\{x\colon f(x) \leqslant a\}$ is everywhere dense in $(x_0, x_0 + \delta)$. Since f satisfies Darboux property, there is $x_1 \in (x_0, x_0 + \delta)$ such that $f(x_0) > f(x_1) > a$. Let

$$G = [x_0, x_1] \cap \{x\colon f(x) > f(x_1)\}.$$

Then G is non-dense, because if G would be dense in some subinterval of $[x_0, x_1]$, then such a subinterval would contain no point of continuity of f, contradicting the fact that f is of Baire class 1. Also since f has Darboux property, G has no isolated points and $(f(x_1), f(x_0)) \subset f(G)$. Finally, since $G \subset E \cap [x_0, x_0 + \delta)$, G is of measure zero. Let $\{Q\}$ be the collection of all non-degenerate components of $[x_0, x_1] - G$. Then the set $\bar{G} = [x_0, x_1] - \bigcup Q^0$, where Q^0 is the interior of Q relative to $[x_0, x_1]$ and the union extends over all $Q \in \{Q\}$, is perfect in $[x_0, x_1]$. Also since G is of measure zero, $\bar{G}$ is of measure zero. Thus $\bar{G}$ is a perfect set of measure zero and $(f(x_1), f(x_0)) \subset f(\bar{G})$. But this contradicts our hypothesis. Hence we conclude that E intersects every right neighbourhood of x_0 in a set of positive measure. Similarly, E intersects every left neighbourhood of x_0 in a set of positive measure. Hence $E \in \mathscr{M}_2$. Similarly, $\{x\colon f(x) < a\} \in \mathscr{M}_2$. Thus $f \in \mathscr{M}_2$.

Remarks. The converse of Theorem 4 is not true. For let P be the Cantor perfect set in $[0, 1]$ and let f be the Cantor increasing function in $[0, 1]$. Then P is of measure zero and $(0, 1) \subset f(P)$. But f is continuous in $[0, 1]$ and hence $f \in \mathscr{M}_2$.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF BURDWAN
BURDWAN, WEST BENGAL, INDIA

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