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FINITE GENERATORS OF ERGODIC ENDOMORPHISMS

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Let τ be an ergodic endomorphism of the Lebesgue space $(X, \mathscr{B}, \mu)$. For τ invertible Krieger [1] proved that τ has a generator that contains k elements, $e^{h(\tau)} \leqslant k \leqslant e^{h(\tau)}+1$, where $h(\tau)$ denotes the entropy of τ. We want to find some conditions on the existence of a finite 1-sided generator α for τ in the non-invertible case and the upper and lower bounds for the minimal number of elements of α.

THEOREM 1. *An ergodic endomorphism τ has a finite 1-sided generator iff there exists an integer n such that* $\operatorname{card}(\tau^{-1}(x)) \leqslant n$ *a.e. and* $h(\tau, \varepsilon) = h(\tau) < \infty$, *where* $\varepsilon = \{\{x\}: x \in X\}$.

Proof. Krieger [1] proved that every exhaustive σ-algebra of an ergodic invertible measure-preserving transformation T whose entropy $h(T)$ is finite contains a finite generator. Hence, using the same reasoning as in the proof of Rohlin's Theorem 10.11 in [2] we get the thesis.

Remark 1. Since $h(\tau) < \infty$, we may assume without loss of generality that τ is positively measurable and positively nonsingular and that the condition $\operatorname{card}(\tau^{-1}(x)) \leqslant n$ a.e. holds for such transformations throughout this paper.

We assume throughout the rest of the paper that the generators considered are 1-sided. Let $\alpha = \{A_i\}_{i=1}^{k}$ be a generator of τ and let $S = \{1, \ldots, k\}$, where $k \leqslant \infty$ and $\mu(A_i) > 0$ for $i \leqslant k$. Then (X, μ, τ) is isomorphic to (S^N, ν, σ), where σ is the shift on S^N, $N = \{0, 1, \ldots\}$. Put

$$n_\tau = \inf\{n: \operatorname{card}(\tau^{-1}(x)) \leqslant n \text{ a.e.}\}$$

and denote by $|\alpha|$ the number of elements of α.

Remark 2. If α is a generator of τ, then $n_\tau \leqslant |\alpha|$.

Let $C_i = \{j: \nu(Z_{ji}) > 0\}$, where $Z_{ji} = \{(x_i)_0^\infty \in S^N: x_0 = j, x_1 = i\}$. Put

$$k_\nu = \max_{i \in S} \operatorname{card}(C_i) \quad \text{and} \quad \Gamma = \{0, \ldots, k_\nu\}.$$

LEMMA 1. *For every shift-invariant ergodic probability measure ν on S^N there exist a shift-invariant probability measure γ on Γ^N and an isomorphism*

$$\phi\colon (S^N, \nu, \sigma) \to (\Gamma^N, \gamma, \sigma).$$

Proof. Let φ_i, $2 \leqslant i \leqslant k_\nu$, be 1-1 mappings from C_i to $\Gamma - \{0\}$ and let φ_1 be a 1-1 mapping from C_1 to Γ such that $\varphi_1(i_0) = 0$ for some $i_0 \in C_1$. Let

$$Y = \{(x_i)_0^\infty \in S^N\colon\ \nu(Z_{x_{i-1}x_i}) > 0,\ i = 1, 2, \ldots, \text{ and } x_j = i_0,\ x_{j+1} = 1 \text{ for infinitely many } j \in N\}.$$

By the ergodicity of ν we get $\nu(Y) = 1$. We define a mapping $\phi\colon Y \to \Gamma^N$ as follows:

$$\phi\big((x_i)_0^\infty\big) = (y_i)_0^\infty, \quad \text{where } y_i = \varphi_{x_{i+1}}(x_i).$$

Now, ϕ is a Borel mapping which commutes with the shift. We show that ϕ is 1-1. Let $x = (x_i)_0^\infty$ and $z = (z_i)_0^\infty$. Assume that $\phi(x) = \phi(z)$. Let i_1 be the first index such that $x_{i_1} \neq z_{i_1}$. The assumption $\phi(x) = \phi(z)$ implies

$$\varphi_{x_{i_1+n}}(x_{i_1+n-1}) = \varphi_{z_{i_1+n}}(z_{i_1+n-1}) \quad \text{for } n = 1, 2, \ldots$$

By the definition of φ_i, $1 \leqslant i \leqslant k$, we have $x_i \neq z_i$ for $i \geqslant i_1$. From the definition of Y we infer that there exists an index i, $i > i_1$, such that $x_i = i_0$ and $x_{i+1} = 1$. The definition of φ_1 implies that if $0 = \varphi_{x_{i+1}}(x_i) = \varphi_{z_{i+1}}(z_i)$, then $z_i = x_i = i_0$ and $x_{i+1} = z_{i+1} = 1$. Hence $i_1 = \infty$ and $x = z$. Let $\gamma = \phi\nu$. Then the spaces (S^N, ν, σ) and $(\Gamma^N, \gamma, \sigma)$ are isomorphic.

Let $\alpha = \{A_1, A_2, \ldots\}$ be a partition of X. We say that α is a *Markovian partition* of X if $\tau(A_i)$ is the union of some elements of α for $i = 1, 2, \ldots$ Observe that if a generator α is Markovian, then $k_\nu = n_\tau$. Hence, using Lemma 1, we get

LEMMA 2. *If an ergodic endomorphism τ has a countable Markovian generator, then it has a generator α such that $n_\tau \leqslant |\alpha| \leqslant n_\tau + 1$.*

We apply Lemma 2 to a class of topological dynamical systems. Let X be a compact metric space with a metric d and let τ be a local homeomorphism of X on X. Assume that τ is *expanding* for this metric, i.e. there are numbers $\lambda > 1$ and $\varepsilon > 0$ such that $d(\tau(x), \tau(y)) \geqslant \lambda d(x, y)$ for $d(x, y) < \varepsilon$. By the results of [3], there exists a finite Markovian partition for τ. This partition is a generator for an ergodic locally positive measure μ. By Lemma 2 we get

THEOREM 2. *Every expanding local homeomorphism τ with ergodic locally positive measure has a finite generator α such that $n_\tau \leqslant |\alpha| \leqslant n_\tau + 1$.*

Remark 3. In general, we cannot replace the inequality in Theorem 2 by the equality $|\alpha| = n_\tau$.

Proof. Let $h(\tau) = \log n_\tau$. It is easy to see that if τ is not a 1-sided Bernoulli shift, then $|\alpha| \geqslant n_\tau + 1$ for every generator α.

REFERENCES

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