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A PROPERTY OF QUASI-COMPLEMENTS

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A closed linear subspace Y of the normed linear space X is said to be *quasi-complemented in* X if there exists a closed linear subspace Z such that $Y \cap Z = \{0\}$ and $Y+Z$ is dense in X. In this case Z is called a *quasi-complement of* Y. If Z is closed, $Y \cap Z = \{0\}$ and $X = Y+Z$, then Z is called a *complement of* Y. Not every closed subspace of a normed linear space has a complement. However, every closed subspace of a separable normed linear space has a quasi-complement by a result of Mackey [2]. Hence, this is one property of quasi-complements which does not hold for complements.

The purpose of this note is to give another nice property of quasi-complements which is not necessarily shared by complements and which has not been mentioned in the literature. If Y and Z are closed subspaces of a Banach space X such that $Y \cap Z = \{0\}$, then, even if Y is complemented in X, there does not necessarily exist a complement Z_1 of Y such that $Z \subset Z_1$. For example, in $X = c_0$, let Y be the set of all sequences of the form $(x_1, 0, x_3, 0, \ldots)$ and let Z be the set of all sequences of the form $(y_1, 2^{-1}y_1, y_2, 2^{-2}y_2, \ldots)$. Then Y and Z are closed linear subspaces of c_0, Y is complemented in c_0, and $Y \cap Z = \{0\}$. However, there is no complement of Y in c_0 which contains Z. In other words, complements of a subspace do not necessarily occur homogeneously throughout X. We now show, however, that quasi-complements do occur more homogeneously than complements.

THEOREM 1. *Let X be a separable normed linear space and let Y be a closed linear subspace of X. If Z is a closed linear subspace of X such that $Y \cap Z = \{0\}$, then Y has a quasi-complement Z_1 such that $Z \subset Z_1$.*

Proof. The normed linear space X/Z is separable. Let $Q\colon X \to X/Z$ denote the quotient map g. By Mackey's theorem, there is a quasi-complement Y_1 of $\overline{Q(Y)}$ in X/Z. Thus $Y_1 \cap \overline{Q(Y)} = \{0\}$ and $Y_1 + \overline{Q(Y)}$ is dense in X/Z. Now, $Q^{-1}(Y_1)$ is closed in X and $Z \subset Q^{-1}(Y_1)$. Let $Z_1 = Q^{-1}(Y_1)$. If $x \in Y \cap Z_1$, then $Q(x) \in Y_1 \cap Q(Y) = \{0\}$, so that $x \in Z$. Since $Y \cap Z = \{0\}$, we have $x = 0$. Consequently, $Y \cap Z_1 = \{0\}$.

It remains only to show that $Y+Z_1$ is dense in X. Let $x \in X$ and let U be an open set containing x. Then $Q(U)$ is an open set in X/Z which contains $Q(x)$. Since $Y_1+\overline{Q(Y)}$ (and hence $Y_1+Q(Y)$) is dense in X/Z, there exists a $z_1 \in Z_1$ and a $y \in Y$ such that $Q(z_1)+Q(y) \in Q(U)$. It follows that $z_1+y_1 \in Z+U$, so that there exists a $z \in Z$ with $y+(z+z_1) \in U$. Noting that $z+z_1 \in Z_1$, the proof is complete.

Remark. Weber [3] has generalized Mackey's result by showing that every closed linear subspace of a separable metrizable locally convex space has a quasi-complement. It is then clear that Theorem 1 holds when X is such a space.

A Banach space is said to be *weakly compactly generated* (*WCG*) if there is a weakly compact set A in X such that X is the closed linear span of A. In particular, every reflexive Banach space is (WCG). Lindenstrauss [1] has shown that if X is a (WCG) Banach space and Y is a closed linear subspace of X which is also (WCG), then Y is quasi-complemented in X. Since continuous linear images of (WCG) spaces are (WCG), Lindenstrauss' theorem, together with the same argument as in Theorem 1, yields the following

THEOREM 2. *Let X be a (WCG) Banach space and let Y be a closed linear subspace of X which is also (WCG). If Z is a closed linear subspace of X such that $Y \cap Z = \{0\}$, then Y has a quasi-complement Z_1 such that $Z \subset Z_1$.*

REFERENCES

[1] J. Lindenstrauss, *Weakly compact sets — their topological properties and the Banach spaces they generate,* Proceedings of the Symposium on Infinite Dimensional Topology, L. S. U. 1967.

[2] G. Mackey, *A note on a theorem of Murray,* Bulletin of the American Mathematical Society 52 (1946), p. 322-325.

[3] J. K. Weber, Jr., *Quasi-complements in separable locally convex spaces,* Abstract 682-46-11, Notices of the American Mathematical Society 18 (1971), p. 178.

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