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# *MULTIPLICITY TYPE AND SUBALGEBRA STRUCTURE IN INFINITARY UNIVERSAL ALGEBRAS*

BY

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**Introduction.** This paper is a continuation of the author's work [2] on multiplicity type and subalgebra structure. A prior reading of [2] is recommended for an understanding of the goals, as well as some of the techniques, of the present paper. A multiplicity type is a transfinite sequence of cardinals, not all zero. Given a universal algebra $\langle A; F\rangle$ and a multiplicity type $\mu = \langle \mu_0, \mu_1, \ldots, \mu_a, \ldots\rangle_{a<\gamma}$, we say(*) that $\langle A; F\rangle$ has *multiplicity type* $\mu$, and we write $\langle A; F\rangle \epsilon K(\mu)$, provided $\gamma > \sup\{a|\text{There is an } a\text{-ary operation in } F\}$ and, for each $a<\gamma$, $\mu_a = |\{f \epsilon F| f \text{ is } a\text{-ary}\}|$. For a multiplicity type $\mu$, let $T(\mu) = \{S(A;F)| \langle A; F\rangle \epsilon K(\mu)\}$, where $S(A;F)$ denotes the family of all carrier-sets of non-void subalgebras of $\langle A; F\rangle$. For multiplicity types $\mu$ and $\mu'$ define a quasi-ordering $\leqslant$ and an equivalence $\equiv$ as follows: $\mu \leqslant \mu'$ if $T(\mu) \subseteq T(\mu')$ and $\mu \equiv \mu'$ if $T(\mu) = T(\mu')$. We shall characterize the quasi-ordering among certain multiplicity types which we call "standard", and we shall exhibit a class $\mathcal{N}$ of multiplicity types "in normal form", whereby every standard multiplicity type will be equivalent to a unique member of $\mathcal{N}$, and the ordering among members of $\mathcal{N}$ will be precisely the pointwise ordering. Further, every member of $\mathcal{N}$ will be greatest in the pointwise ordering of all standard multiplicity types equivalent to it, and $\mathcal{N}$ will contain the class $\mathcal{N}^*$, exhibited in [2], of "maximal normal forms" for multiplicity types of finitary algebras only.

As this paper is an extension of [2], some results of [2] will be adopted here without proof. Related material on subalgebra structure can be found in [1] for finitary algebras and in [3] for infinitary algebras. Further background material can be found in the monograph [5] of Pierce and the book [4] of Grätzer. Notations used here are those of [4].

(*) Some of these results are taken from the author's doctoral thesis 1967, supervised by Prof. G. Grätzer at The Pennsylvania State University. Futher research was supported by the National Science Foundation Under Grant GP-8725.

**1. Preliminaries.** Most of the results of this section are ready generalizations of results in [2], and are therefore stated without proof. First some definitions are needed, largely to establish notation. The symbols $\mu$ and $\mu'$ will always denote multiplicity types.

Definition. We say that $\mu'$ *accepts* $\mu$ provided $\mu_0 = 0$ implies $\mu'_0 = 0$. If each accepts the other, then $\mu$ and $\mu'$ are termed *compatible.*

Definition. Let $\varkappa$ be an ordinal and $\mathfrak{m}$ a non-zero cardinal. We shall denote by $\varepsilon_\varkappa(\mathfrak{m})$ the multiplicity type having $\mathfrak{m}$ as its $\varkappa$th entry and zeroes everywhere else. For simplicity, $\varepsilon_\varkappa(1)$ will be denoted by $\varepsilon_\varkappa$.

Definition. Let $\{\mu^i|i\in I\}$ be a set of multiplicity types. We write $\mu = \sum(\mu^i|i\in I)$ if $\mu_a = \sum(\mu^i_a|i\in I)$ for each $a$. The symbol $+$ will be used in the usual fashion when $|I| = 2$.

Definition. The *extent* of $\mu$, an ordinal denoted by $e(\mu)$, is defined by $e(\mu) = \sup\{a|\mu_a \neq 0\}$. If $\mu_{e(\mu)} \neq 0$, we say that $\mu$ is *closed.* We define an ordinal $e^*(\mu)$ to be $e(\mu)+1$ if $\mu$ is closed, and $e(\mu)$ otherwise. We set $l(\mu) = \overline{\overline{e(\mu)}}$ and call $l(\mu)$ the *length* of $\mu$. The ordinal $b(\mu)$ is defined as follows. If $\mu$ is not closed and $l(\mu)$ is a regular cardinal, let $b(\mu) = e(\mu)$. In all other cases, let $b(\mu)$ be the initial ordinal whose cardinality is the cardinal successor of $l(\mu)$.

Definition. We say that $\mu$ is *initial* provided $\mu_a \neq 0$ implies $a$ is an initial ordinal. An initial multiplicity type is called *standard* if it is closed, or if its length is a regular cardinal. Finally, $\mu$ is called *infinitary* if there is some $a \geqslant \omega$ such that $\mu_a \neq 0$.

LEMMA 1.1.

(i) *If* $\mu \leqslant \mu'$, *then* $\mu'$ *accepts* $\mu$.

(ii) *If* $\mu'$ *accepts* $\mu$ *and* $\mu_a \leqslant \mu'_a$ *for all* $a$, *then* $\mu \leqslant \mu'$.

(iii) $\varkappa < \lambda$ *implies* $\varepsilon_\varkappa(\mathfrak{m}) \leqslant \varepsilon_\lambda(\mathfrak{m})$ *for all* $\mathfrak{m}$.

(iv) *Let* $a$ *and* $\beta$ *be ordinals with* $a < \beta$, *and let* $\{\mathfrak{m}_\gamma|a \leqslant \gamma < \beta\}$ *be a set of cardinals. Then*

$$\varepsilon_a\Big(\sum(\mathfrak{m}_\gamma|a \leqslant \gamma < \beta)\Big) \leqslant \sum\big(\varepsilon_\gamma(\mathfrak{m}_\gamma)|a \leqslant \gamma < \beta\big).$$

(v) *If* $\mu^i$ *and* $\nu^i$ $(i\in I)$ *are multiplicity types with* $\mu^i \leqslant \nu^i$ *for each* $i\in I$, *then*

$$\sum(\mu^i|i\in I) \leqslant \sum(\nu^i|i\in I).$$

By a *restricted closure system* we shall mean a family $\mathfrak{A}$ of subsets of a set $A$, having the property that whenever $\mathfrak{B} \subseteq \mathfrak{A}$ and $\bigcap(X|X\in\mathfrak{B}) \neq \emptyset$, then $\bigcap(X|X\in\mathfrak{B})\in\mathfrak{A}$. If $\emptyset \neq B \subseteq A$, then the *closure* of $B$, denoted $[B]$, is defined to be $\bigcap(X|X\in\mathfrak{A}, X \supseteq B)$. It is easily seen that $S(A;F)$ is a restricted closure system for any algebra $\langle A;F\rangle$. Moreover, we have

THEOREM 1.1. *Let* $\mathfrak{A}$ *be a restricted closure system of subsets of* $A$, *and let* $\mathfrak{n}$ *be a non-zero cardinal. The following are equivalent.*

(i) $\mathfrak{A} \in T(\mu)$ *for some* $\mu$ *of length* $\mathfrak{n}$.

(ii) *If* $\emptyset \neq B \subseteq A$ *and* $B = \bigcup([C] | C \subseteq B, 0 < |C| \leqslant \mathfrak{n})$, *then* $B \in \mathfrak{A}$.

COROLLARY. *If* $\mu \leqslant \mu'$ *and* $e(\mu)$ *is an initial ordinal, then* $l(\mu) \leqslant l(\mu')$.

LEMMA 1.2. *Every multiplicity type is equivalent to an initial multiplicity type. Specifically,* $\mu \equiv \varrho\mu$, *where* $(\varrho\mu)_a = \sum(\mu_\varkappa | \bar{\varkappa} = \bar{a})$ *if* $a$ *is initial, and* $(\varrho\mu)_a = 0$ *otherwise.*

The following lemma provides a useful construction:

LEMMA 1.3. *Suppose* $\mu$ *is a multiplicity type and* $a$ *an ordinal with* $0 < \bar{a} < \mu_a$. *Let* $A$ *be a set with* $|A| = \mu_a$, *let* $C \subseteq A$ *with* $|C| = \bar{a}$, *and let* $p \in A \setminus C$ *Let* $\mathfrak{A} = \{A\} \cup \{B | C \nsubseteq B \subseteq A\}$, *and let* $\mathfrak{A}_p = \{B \cup \{p\} | B \in \mathfrak{A}\}$. *If* $\mu_0 = 0$, *then* $\mathfrak{A} \in T(\mu)$, *and if* $\mu_0 \neq 0$, *then* $\mathfrak{A}_p \in T(\mu)$.

The following lemma is well-known (for a proof see [3] or [5]):

LEMMA 1.4. *Let* $B \subseteq A$, *where* $\langle A; F\rangle \in K(\mu)$ *and* $\mu$ *is initial.*

(i) $[B] = \bigcup(B_a | a < b(\mu))$, *where the sets* $B_a$ *are defined as follows:* $B_0 = B$. *If* $a = \beta + 1$, *then*

$$B_a = B_\beta \cup \bigcup\Big(\bigcup\big(f(B_\beta^\lambda) | f \in F \text{ and } f \text{ is } \lambda\text{-ary}\big) | \lambda < e^*(\mu)\Big).$$

*If* $a$ *is a limit ordinal, then* $B_a = \bigcup(B_\gamma | \gamma < a)$.

(ii) $|[B]| \leqslant (|B| + s(\mu) + \aleph_0)^{l(\mu)}$, *where* $s(\mu) = \sum(\mu_a | \mu_a \neq 0)$.

**2. Technical lemmas.** The following two lemmas provide a kind of calculus for multiplicity types to facilitate the proof of our main result, Theorem 3.1, which will establish a maximal form for every standard infinitary multiplicity type.

LEMMA 2.1. (i) *Let* $\lambda$ *and* $\varkappa$ *be ordinals such that* $\lambda$ *is infinite and* $\lambda \geqslant \varkappa$, *and let* $\mathfrak{m}$ *be a non-zero cardinal. Then* $\varepsilon_\varkappa(\mathfrak{m}^{\bar{\lambda}}) \leqslant \varepsilon_\varkappa(\mathfrak{m}) + \varepsilon_\lambda$.

(ii) *If* $\lambda$ *is an infinite ordinal, then* $\varepsilon_\lambda(2^{\bar{\lambda}}) \equiv \varepsilon_\lambda$.

Proof. (i) Let $\mathfrak{A} \in T(\varepsilon_k(\mathfrak{m}^{\bar{\lambda}}))$, where $\mathfrak{A}$ is a restricted closure system of subsets of the set $A$, and let $a$ be an ordinal of power $\mathfrak{m}$. For each $B \subseteq A$ with $0 < |B| \leqslant \bar{\varkappa}$, choose a set $\tilde{B}$ as follows.

If $|[B]| \leqslant \mathfrak{m}$, let $\tilde{B} = [B]$.

If $|[B]| > \mathfrak{m}$, let $\tilde{B}$ be any set such that $|\tilde{B}| = \mathfrak{m}$ and $\tilde{B} \subseteq [B]$.

Enumerate $\tilde{B}$ as $\{\tilde{b}_0, \tilde{b}_1, \ldots, \tilde{b}_\delta, \ldots\}$, $\delta < a$, with repetition if necessary. For each $\delta < a$, define $g_\delta^B$: $B^{(\varkappa)} \to \tilde{B}$ by $g_\delta^B(x) = \tilde{b}_\delta$ for all $x \in B^{(\varkappa)}$, where $B^{(\varkappa)}$ denotes the set of all onto members of $B^\varkappa$.

If $\mathfrak{m} = 1$, the lemma is trivial, so assume $\mathfrak{m} > 1$. Then by (ii) of Lemma 1.4, $|[B]| \leqslant \mathfrak{m}^{\bar{\lambda}}$. Therefore we let $f^B$ be any map of $\tilde{B}^\lambda$ onto $[B]$.

For each $\delta < a$ define $g_\delta$: $A^\varkappa \to A$ as follows.

If $x = \langle x_0, \ldots, x_\beta, \ldots\rangle \in A^\varkappa$, let $B = \{x_0, \ldots, x_\beta, \ldots\}$ and $g_\delta(x) = g_\delta^B(x)$.

Define $f^*$: $A^{\varkappa+\lambda} \to A$ as follows.

If $z = \langle x_0, \ldots, x_\beta, \ldots; y_0, \ldots, y_\gamma, \ldots\rangle \in A^{\varkappa+\lambda}$, let $B = \{x_0, \ldots, x_\beta, \ldots\}$ and let

$$f^*(z) = \begin{cases} f^B(y_0, \ldots, y_\gamma, \ldots) & \text{if } \{y_0, \ldots, y_\gamma, \ldots\} \subseteq \tilde{B}, \\ x_0 & \text{otherwise.} \end{cases}$$

Finally, noting that $\overline{\varkappa+\lambda} = \bar{\lambda}$, we let $f$ be any $\lambda$-ary function with the property that $S(A;f) = S(A;f^*)$.

Letting $\mathfrak{B} = S(A;f, \{g_\delta|\delta < a\})$, we have $\mathfrak{B} \in T(\varepsilon_\varkappa(\mathfrak{m}) + \varepsilon_\lambda)$, and $\mathfrak{B} = S(A;f^*, \{g_\delta|\delta < a\})$. We show that $\mathfrak{A} = \mathfrak{B}$.

Let $X \in \mathfrak{A}$, $x_\beta, y_\gamma \in X$, $\beta < \varkappa$, $\gamma < \lambda$, and let $B = \{x_0, \ldots, x_\beta, \ldots\}$.

For each $\delta < a$, $g_\delta(x_0, \ldots, x_\beta, \ldots) = g_\delta^B(x_0, \ldots, x_\beta, \ldots) \in \tilde{B} \subseteq [B] \subseteq X$; and $f^*(x_0, \ldots, x_\beta, \ldots; y_0, \ldots, y_\gamma, \ldots) \in [B] \subseteq X$. Thus $X$ is closed under $f^*$ and $g_\delta$, whence $X \in \mathfrak{B}$, so $\mathfrak{A} \subseteq \mathfrak{B}$.

Now let $X \in \mathfrak{B}$. To show that $X \in \mathfrak{A}$ it suffices by Theorem 1.1 to show that $X = \bigcup([B]|B \subseteq X, 0 < |B| \leqslant \bar{\varkappa})$.

So let $B \subseteq X$, $0 < |B| \leqslant \bar{\varkappa}$.

First we show that $\tilde{B} \subseteq X$. Let $\tilde{b}_\delta \in \tilde{B}$. Then $\tilde{b}_\delta \in g_\delta^B(B^{(\varkappa)}) \subseteq g_\delta(B^\varkappa) \subseteq X$, so $\tilde{B} \subseteq X$. Now if $b \in [B]$, then there are $y_\gamma \in \tilde{B}$, $\gamma < \lambda$, such that $b = f^B(y_0, \ldots, y_\gamma, \ldots)$. If $\{b_0, \ldots, b_\beta, \ldots\}_{\beta<\varkappa}$ is an enumeration of $B$, then $b = f^B(y_0, \ldots, y_\gamma, \ldots) = f^*(b_0, \ldots, b_\beta, \ldots, y_0, \ldots, y_\gamma, \ldots) \in X$, and this completes the proof of (i).

(ii) Setting $\lambda = \varkappa$ in (i), we have $\varepsilon_\lambda(2^{\bar{\lambda}}) \leqslant \varepsilon_\lambda(3)$, so it suffices to show $\varepsilon_\lambda(3) \leqslant \varepsilon_\lambda$, and clearly we need show only that $\varepsilon_\lambda(2) \leqslant \varepsilon_\lambda$; equivalently, we show that $\varepsilon_\lambda(2) \leqslant \varepsilon_{1+\lambda}$. To this end, let $\mathfrak{A} \in T(\varepsilon_\lambda(2))$; $\mathfrak{A} = S(A;f_1,f_2)$, where $f_1$ and $f_2$ are $\lambda$-ary. Define $f$: $A^{1+\lambda} \to A$ as follows, for all $z = \langle x; x_0, \ldots, x_\gamma, \ldots\rangle \in A^{1+\lambda}$.

If $z$ is constant and $f_1(x_0, \ldots, x_\gamma, \ldots) = x$, let $f(z) = f_2(x_0, \ldots, x_\gamma, \ldots)$.

In all other cases, define

$$f(z) = \begin{cases} f_1(x_0, \ldots, x_\gamma, \ldots) \text{ if } x = x_0, \\ f_2(x_0, \ldots, x_\gamma, \ldots) \text{ if } x \neq x_0. \end{cases}$$

It is straightforward to verify that $\mathfrak{A} = S(A;f) \in T(\varepsilon_{1+\lambda})$.

LEMMA 2.2. *Let $\mu$ be standard and of infinite length, and let $\{{}^\gamma\mu|\gamma < b(\mu)\}$ be a set of multiplicity types. If ${}^\gamma\mu \leqslant \mu$ for each $\gamma < b(\mu)$, then $\sum({}^\gamma\mu|\gamma < b(\mu)) \leqslant \mu$.*

Proof. By (v) of Lemma 1.1 it suffices to show that $\mu' \leqslant \mu$, where $\mu'_a = \overline{b(\mu)} \cdot \mu_a$ for each $a$.

Case 1. Suppose $\mu$ is closed. Then

$$\mu' \leqslant \sum(\varepsilon_a(2^{l(\mu)} \cdot \mu_a)|\ \mu_a \neq 0)$$
$$\leqslant \sum(\varepsilon_a(\mu_a)|\ \mu_a \neq 0, a < e(\mu)) + \varepsilon_{e(\mu)}(l(\mu) \cdot 2^{l(\mu)} \cdot \mu_{e(\mu)}) \equiv \mu$$

by (ii) of Lemma 2.1.

Case 2. Suppose $\mu$ is not closed and $l(\mu)$ is regular. Then $b(\mu) = e(\mu) = e^*(\mu)$. If $e(\mu) = \omega$, the result follows from the $\mathcal{N}^*$ representation given in [2], so assume $e(\mu) > \omega$. For simplicity we assume also that $\mu_0 = 0$; the other case is analogous.

Let $L = \{\lambda | \omega \leqslant \lambda < e(\mu) \text{ and } \mu_\lambda \neq 0\}$, and choose pairwise disjoint sets $E_\lambda$, $\lambda \in L$, such that $E_\lambda \subseteq \{\gamma | \omega \leqslant \gamma < e(\mu)\}$, $|E_\lambda| = l(\mu)$, and $\inf E_\lambda \geqslant \lambda$. Also, fix some $\beta \in L$.

Using Lemma 1.1, we have

$$\begin{aligned}\mu' &= \mu + \sum\big(\varepsilon_\lambda(l(\mu)) | \mu_\lambda \neq 0\big) \leqslant \mu + \varepsilon_\beta(l(\mu)) + \sum\big(\varepsilon_\lambda(l(\mu)) | \lambda \in L\big)\\ &= \mu + \sum\big(\varepsilon_\lambda(l(\mu)) | \lambda \in L\big) \leqslant \mu + \sum\Big(\sum(\varepsilon_a | a \in E_\lambda) | \lambda \in L\Big)\\ &\leqslant \mu + \sum\big(\varepsilon_\gamma | \omega \leqslant \gamma < e(\mu)\big) \leqslant \mu + \mu.\end{aligned}$$

Now, by Lemma 2.1, $\mu$ is equivalent to a multiplicity type having no finite nonzero entries, whence $\mu + \mu \equiv \mu$, so $\mu' \leqslant \mu$.

**3. Normal form.** We now begin to define the normal form of a standard multiplicity type $\mu$ of infinite length. For every cardinal $\mathfrak{m}$, define $w(\mu, \mathfrak{m})$, the *weight* of $\mu$ with respect to $\mathfrak{m}$, by $w(\mu, \mathfrak{m}) = \sup\{|A| \,|\, \langle A; F\rangle \in K(\mu)$ for some $F$ such that $\langle A; F\rangle$ is generated by a set of cardinality $\mathfrak{m}\}$. The existence of $w(\mu, \mathfrak{m})$ follows from Lemma 1.4. Moreover, $w(\mu, \mathfrak{m})$ is actually the cardinality of an algebra of the sort indicated; it suffices to consider any algebra in $K(\mu)$, generated by a set $B$ of cardinality $\mathfrak{m}$, such that the sets $B_a$ of Lemma 1.4 are as large as possible; this can be achieved by an algebra absolutely freely generated by $B$, a word algebra or algebra of polynomial symbols. (See [4] or [5].) It then follows from Lemma 1.4 that $w(\mu, \mathfrak{m})$ is the sum of the cardinalities of the $B_a$. Thus,

$$w(\mu, \mathfrak{m}) = \sum\big(w_a(\mu, \mathfrak{m}) | a < b(\mu)\big),$$

where $w_a(\mu, \mathfrak{m})$, the *ath partial weight* of $\mu$ with respect to $\mathfrak{m}$, is defined as $w_0(\mu, \mathfrak{m}) = \mathfrak{m}$, and if $a = \beta + 1$, then

$$w_a(\mu, \mathfrak{m}) = w_\beta(\mu, \mathfrak{m}) + \sum\big(\mu_\lambda \cdot w_\beta(\mu, \mathfrak{m})^{\bar{\lambda}} | \lambda < e^*(\mu)\big).$$

If $a$ is a limit ordinal, then $w_a(\mu, \mathfrak{m}) = \sum\big(w_\gamma(\mu, \mathfrak{m}) | \gamma < a\big)$.

Now, for each $\varkappa < e^*(\mu)$, define the multiplicity type $\mu^\varkappa$ by $\mu^\varkappa_\lambda = 0$ for $\lambda < \varkappa$ and $\mu^\varkappa_\lambda = \mu_\lambda$ for $\lambda \geqslant \varkappa$.

Finally, define the multiplicity type $\mu^*$ by $\mu^*_0 = 0$ if $\mu_0 = 0$, and $\mu^*_\varkappa = w(\mu^\varkappa, \bar{\varkappa})$ if $0 < \varkappa < e^*(\mu)$ or if $\varkappa = 0$ and $\mu_0 \neq 0$.

THEOREM 3.1. *If $\mu$ is standard and infinitary, then $\mu \equiv \mu^*$.*

Proof. Clearly $\mu \leqslant \mu^*$, so we show $\mu^* \leqslant \mu$. Define for each $a < e^*(\mu)$ the multiplicity type $\mu[a]$ by $\mu[a]_0 = 0$ if $\mu_0 = 0$, and $\mu[a]_\varkappa = w_a(\mu^\varkappa, \bar{\varkappa})$ if $0 < \varkappa < e^*(\mu)$ or if $\varkappa = 0$ and $\mu_0 \neq 0$. For the sake of simplicity, we assume for the remainder of the proof that $\mu_0 = 0$. We shall show that $\mu[a] \leqslant \mu$ for each $a$.

First consider $\mu[0]$. Using Lemmas 1.1, 2.1 and 2.2, we have

$$\mu[0] = \sum\big(\varepsilon_\varkappa(\bar{\varkappa})|\varkappa < e^*(\mu)\big) \leqslant \varepsilon_\omega(\aleph_0) + \sum\big(\varepsilon_\varkappa(\bar{\varkappa})|\omega \leqslant \varkappa < e^*(\mu)\big)$$
$$= \sum\big(\varepsilon_\varkappa(\bar{\varkappa})|\omega \leqslant \varkappa < e^*(\mu)\big) \equiv \sum\big(\varepsilon_\varkappa|\omega \leqslant \varkappa < e^*(\mu)\big) \leqslant \mu.$$

Now suppose $\alpha = \beta+1$ and $\mu[\beta] \leqslant \mu$. To show that $\mu[\alpha] \leqslant \mu$, we verify the following statements.

STATEMENT 1. *$\varkappa \leqslant \lambda < e^*(\mu)$ implies $w_\beta(\mu^\varkappa, \bar{\varkappa}) \leqslant w_\beta(\mu^\lambda, \bar{\lambda})$.*

The proof is by a trivial transfinite induction on $\beta$.

STATEMENT 2. *If $\lambda, \varkappa < e^*(\mu)$, then $\varepsilon_\lambda(\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}) \leqslant \mu$.*

We distinguish three cases:

Case 1. $\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}} < \aleph_0$.

Choose some $\delta \geqslant \omega, \lambda$ such that $\mu_\delta \neq 0$. Then

$$\varepsilon_\lambda\big(\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}\big) \leqslant \varepsilon_\delta(\aleph_0) \equiv \varepsilon_\delta \leqslant \mu.$$

Case 2. $\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}} \geqslant \aleph_0$ and $\mu_\lambda \geqslant w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}$.

Then $\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}} = \mu_\lambda$ and so $\varepsilon_\lambda\big(\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}\big) = \varepsilon_\lambda(\mu_\lambda) \leqslant \mu$.

Case 3. $\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}} \geqslant \aleph_0$ and $\mu_\lambda \leqslant w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}$.

Then $\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}} = w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}$ and we consider four subcases:

*Subcase* 3a. $\lambda \leqslant \varkappa < \omega$.

Then $\varepsilon_\lambda\big(w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}\big) = \varepsilon_\lambda\big(w_\beta(\mu^\varkappa, \bar{\varkappa})\big) \leqslant \varepsilon_\varkappa\big(w_\beta(\mu^\varkappa, \bar{\varkappa})\big) \leqslant \mu[\beta] \leqslant \mu$.

*Subcase* 3b. $\lambda \leqslant \varkappa$ and $\varkappa \geqslant \omega$.

Using (i) of Lemma 2.1,

$$\varepsilon_\lambda\big(w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}\big) \leqslant \varepsilon_\varkappa\big(w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\varkappa}}\big) \equiv \varepsilon_\varkappa\big(w_\beta(\mu^\varkappa, \bar{\varkappa})\big) \leqslant \mu[\beta] \leqslant \mu.$$

*Subcase* 3c. $\varkappa \leqslant \lambda < \omega$.

Using Statement 1,

$$\varepsilon_\lambda\big(w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}\big) = \varepsilon_\lambda\big(w_\beta(\mu^\varkappa, \bar{\varkappa})\big) \leqslant \varepsilon_\lambda\big(w_\beta(\mu^\lambda, \bar{\lambda})\big) \leqslant \mu[\beta] \leqslant \mu.$$

*Subcase* 3d. $\varkappa \leqslant \lambda$ and $\lambda \geqslant \omega$.

Using Statement 1 and (i) of Lemma 2.1,

$$\varepsilon_\lambda\big(w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}\big) \equiv \varepsilon_\lambda\big(w_\beta(\mu^\varkappa, \bar{\varkappa})\big) \leqslant \varepsilon_\lambda\big(w_\beta(\mu^\lambda, \bar{\lambda})\big) \leqslant \mu[\beta] \leqslant \mu,$$

and so Statement 2 is proved.

Summing on $\lambda$ in Statement 2 and applying Lemma 2.2,

$$\sum\big(\varepsilon_\lambda(\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}})|\lambda < e^*(\mu)\big) \leqslant \mu.$$

Now,

$$\mu[\alpha]_\varkappa \leqslant w_\alpha(\mu^\varkappa, \bar{\varkappa}) \quad \text{(with equality except for } \varkappa = 0)$$
$$= w_\beta(\mu^\varkappa, \bar{\varkappa}) + \sum\big(\mu_\lambda^\varkappa \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}}|\lambda < e^*(\mu)\big)$$
$$= w_\beta(\mu^\varkappa, \bar{\varkappa}) + \sum\big(\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\lambda}|\varkappa \leqslant \lambda < e^*(\mu)\big)$$
$$= \sum\big(\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\lambda}|\varkappa \leqslant \lambda < e^*(\mu)\big),$$

and so, by (iv) of Lemma 1.1,

$$\varepsilon_\varkappa(\mu[a]_\varkappa) \leqslant \sum\left(\varepsilon_\lambda(\mu_\lambda \cdot w_\beta(\mu^\varkappa, \bar{\varkappa})^{\bar{\lambda}})|\varkappa \leqslant \lambda < e^*(\mu)\right) \leqslant \mu.$$

Now, summing on $\varkappa$,

$$\mu[a] = \sum\left(\varepsilon_\varkappa(\mu[a]_\varkappa)|\varkappa < e^*(\mu)\right) \leqslant \mu$$

by Lemma 1.2.

Finally, we consider the case where $a$ is a limit ordinal and $\mu[\gamma] \leqslant \mu$ for all $\gamma < a$. Since we are assuming $\mu_0 = 0$, we have

$$\mu[a]_0 = 0 = \mu[\gamma]_0 \quad \text{for all } \gamma < a.$$

In all other cases, we have

$$\mu[a]_\varkappa = w_a(\mu^\varkappa, \bar{\varkappa}) = \sum\left(w_\gamma(\mu^\varkappa, \bar{\varkappa})|\gamma < a\right),$$

and so

$$\mu[a] = \sum(\mu[\gamma]|\gamma < a) \leqslant \mu$$

by Lemma 1.2.

Thus we have proved that $\mu[a] \leqslant \mu$ for all $a$.

Since

$$w(\mu^\varkappa, \bar{\varkappa}) = \sum\left(w_a(\mu^\varkappa, \bar{\varkappa})|a < b(\mu)\right),$$

we have

$$\mu^* = \sum\left(\mu[a]|a < b(\mu)\right) \leqslant \mu$$

by Lemma 1.2, and the theorem is proved.

THEOREM 3.2. *Let $\mu$ and $\mu'$ be compatible and suppose $\mu'$ is initial and $\mu$ is standard and infinitary. Then $\mu' \leqslant \mu$ if and only if $\mu'_\varkappa \leqslant \mu^*_\varkappa$ for each $\varkappa$.*

Proof. The "if" statement is immediate from Theorem 3.1. Conversely, suppose $\mu' \leqslant \mu$. If $\mu'_0 \neq 0$, then $\mu_0 \neq 0$ because $\mu'$ accepts $\mu$, so choose any set $A$ with $|A| = \mu'_0$ and make each $a \in A$ the value of a nullary operation, thereby obtaining $\{A\} \in T(\mu')$. Then $\mu' \leqslant \mu$ implies $\{A\} \in T(\mu)$ and, since $A = [\emptyset]$, we have $\mu'_0 = |A| \leqslant w(\mu, 0) = w(\mu^0, 0) = \mu^*_0$.

Now let $\varkappa > 0$ such that $\mu'_\varkappa > 0$. If $\mu'_\varkappa \leqslant \bar{\varkappa}$, then $\mu'_\varkappa \leqslant \mu^*_\varkappa$ is immediate, so suppose $\bar{\varkappa} < \mu'_\varkappa$, and consider $C \subseteq A$, where $A$ is a set of cardinality $\mu'_\varkappa$ and $|C| = \bar{\varkappa}$. By Lemma 1.3 we have $\mathfrak{A} \in T(\mu')$ if $\mu'_0 = 0$, where $\mathfrak{A} = \{A\} \cup \{B | C \nsubseteq B \subseteq A\}$. We assume $\mu'_0 = 0$; if not, simply choose $p \in A \setminus C$ and use in place of $\mathfrak{A}$ the family $\mathfrak{A}_p$ of Lemma 1.3.

Since $\mu' \leqslant \mu$, we have $\mathfrak{A} = S(A; F)$ for some $\langle A; F\rangle \in K(\mu)$. Let $\bar{F} = \{f | f \in F \text{ and } f \text{ is } \lambda\text{-ary for some } \lambda \geqslant \varkappa\}$. Note that $\bar{F} \neq \emptyset$ because $\mu'$ is initial and $l(\mu') \leqslant l(\mu)$. Let $\bar{A}$ denote the carrier-set of the subalgebra generated by $C$ in the algebra $\langle A; \bar{F}\rangle$. It is easy to verify that $\bar{A}$ is closed under $F\setminus\bar{F}$, whence $\bar{A} = A$. Since $\langle A; \bar{F}\rangle \in K(\mu^{\varkappa})$, we have, by definition of $w(\mu^{\varkappa}, \bar{\varkappa})$, that $\mu'_{\varkappa} = |\bar{A}| \leqslant w(\mu^{\varkappa}, \bar{\varkappa}) = \mu^*_{\varkappa}$, and the theorem is proved.

It now follows that $(\varrho\mu^*)_{\varkappa} = \mu^*_{\varkappa}$ for initial ordinals $\varkappa$, where $\varrho\mu^*$ is the initial multiplicity type defined in Lemma 1.2. Therefore, Theorem 3.2 still holds if $\mu^*$ is replaced by $\varrho\mu^*$. Recalling the class $\mathscr{N}^*$, defined in [2], of maximal normal forms for multiplicity types which are not infinitary, we now define normal form for all standard multiplicity types.

Definition. Let $\mathscr{N} = \mathscr{N}^* \cup \{\varrho\mu^* | \mu$ is standard and infinitary$\}$.

COROLLARY. *Every standard multiplicity type $\mu$ is equivalent to a unique member of $\mathscr{N}$, which is greatest in the pointwise ordering of all standard multiplicity types equivalent to $\mu$. Among members of $\mathscr{N}$, the ordering is precisely the pointwise ordering.*

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