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SETS OF INTERPOLATION AND SMALL p SETS

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In this paper G is a compact nondiscrete abelian group with character group Γ and $M(G)$ is the usual convolution algebra of Borel measures on G. The Fourier–Stieltjes transform of the measure $\mu \in M(G)$ is the function $\hat{\mu}$ defined on Γ by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma(x)}\, d\mu(x).$$

We designate by $M_a(G)$ those $\mu \in M(G)$ absolutely continuous with respect to Haar measure on G and $M_s(G)$ those $\mu \in M(G)$ singular with respect to Haar measure. Denote by $M_d(G) \subset M(G)$ the algebra of discrete measures. A subset A of Γ is said to satisfy (*) if

$$(*) \qquad M_a(G)\hat{}\,|_A \subset M_d(G)\hat{}\,|_A.$$

The interpolation condition (*) has been studied by many authors; in particular the reader is referred to [1], [3] and [6].

A subset $S(p)$ of Γ is called a *small p set* if whenever $\operatorname{supp}\hat{\mu} \subset S(p)$ the pth fold convolution $\mu^p \in M_a(G)$. Small 1 sets are called *Riesz sets*. It is known that the union of a small p set and a Sidon set is a small p set [7]. It is also known that the union of a Riesz set and a Rosenthal set is a Riesz set, see [2]. For examples of Riesz sets and small p sets the reader is referred to [4] and [8]. See also [5] in this connection.

Throughout this paper F^a designates the set of accumulation points of $F \subset \Gamma$ where Γ is endowed with the relative topology of its Bohr-compactification. We state and prove our main result.

THEOREM. *Let $S(p)$ be a small p set and let A satisfy (*). Then $S(p) \cup A$ is small p^2 set.*

Proof. We shall assume without loss of generality that $S(p) \cap A = \emptyset$. Let $\mu \in M(G)$ with

$$(1) \qquad \operatorname{supp}\hat{\mu} \subset S(p) \cup A.$$

Put $\nu = \mu^p$ and set $\nu = \nu_a + \nu_s$ where $\nu_a \in M_a(G)$ and $\nu_s \in M_s(G)$. It follows quite easily from the methods of Y. Meyer [4] that (1) implies:

$$\operatorname{supp} \hat{\nu}_s \subset \bar{A}. \tag{2}$$

Here $\bar{A}$ is the closure of A in Γ endowed with the relative topology of its Bohr-compactification.

Let $\gamma_0 \in A$. Choose $\varrho \in M_a(G)$ such that $\hat{\varrho}(\gamma_0) = 1$. Since A satisfies $(*)$ there is a $\mu_d \in M_d(G)$ such that

$$\hat{\mu}_d = \hat{\varrho} \quad \text{on } A. \tag{3}$$

It now follows from (1) and (3) that

$$\operatorname{supp} (\mu_d * \mu - \varrho * \mu)\hat{} \subset S(p). \tag{4}$$

Inasmuch as $S(p)$ is a small p set and $M_a(G)$ is an ideal in $M(G)$ we gather from (4) that

$$\mu_d^p * \nu_s \in M_a(G). \tag{5}$$

It follows immediately from (5) that $\mu_d^p * \nu_s = 0$. From (3) we see that $\hat{\nu}_s(\gamma_0) = 0$ and so it is permissible to infer that:

$$\hat{\nu}_s(A) = 0. \tag{6}$$

Notice that by the main result of [6], $A \cap A^a = \emptyset$. Now (2) and (6) together give:

$$\operatorname{supp} \hat{\nu}_s \subset A^a. \tag{7}$$

As a consequence of the main result of [6] there is a measure $\omega \in M(G)$ such that

$$\hat{\omega}(A) = 0, \tag{i}$$

$$\hat{\omega}(A^a) = 1. \tag{ii}$$

Consider the measure

$$\xi = \nu_a - \omega * \nu_a \in M_a(G).$$

Notice first that $\hat{\xi}(\gamma) = \hat{\nu}_a(\gamma)$ for all $\gamma \in A$ by (i). Notice also that $\hat{\nu}(\gamma) = \hat{\nu}_a(\gamma)$ for all $\gamma \in A$ by (6). Thus

$$\hat{\xi}(\gamma) = \hat{\nu}(\gamma) \quad \text{for all } \gamma \in A. \tag{8}$$

Next we see that $\hat{\xi}(\gamma) = 0$ for all $\gamma \in A^a \setminus S(p)$ by appeal to (ii). Since $\hat{\nu}(\gamma) = 0$ for all $\gamma \in A^a \setminus S(p)$ we obtain:

$$\hat{\xi}(\gamma) = \hat{\nu}(\gamma) \quad \text{for all } \gamma \in A^a \setminus S(p). \tag{9}$$

Finally, if $\gamma \notin S(p) \cup A \cup (A^a \setminus S(p))$, then $\hat{\nu}(\gamma) = 0$ therefore $\hat{\nu}_s(\gamma) = -\hat{\nu}_a(\gamma)$. Since $\gamma \notin A^a$ we obtain via (7) that $\hat{\nu}_a(\gamma) = 0$. Thus we conclude that:

$$\hat{\xi}(\gamma) = \hat{\nu}(\gamma), \quad \gamma \notin S(p) \cup A \cup (A^a \setminus S(p)). \tag{10}$$

An easy consequence of (8), (9) and (10) is

$$\hat{\xi}(\gamma) = \hat{\nu}(\gamma) \quad \text{for all } \gamma \notin S(p). \tag{11}$$

Since $S(p)$ is a small p set we gather from (11) that $(\nu - \xi)^p \in M_a(G)$. Our result now follows from the fact that $\nu^p \in M_a(G)$.

COROLLARY. *The union of a Riesz set and any set satisfying* $(*)$ *is again a Riesz set.*

It is an open question as to whether or not every small p set is a Riesz set. It is also apparently unknown if all sets which satisfy the interpolation property $(*)$ are strong Riesz sets. **(P 1299)**

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