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ARCWISE CONNECTED AND LOCALLY ARCWISE CONNECTED SETS

BY

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A well-known theorem of R. L. Wilder says that a continuum (i.e., a connected compact metric space) is hereditarily locally connected if and only if each connected subset of it is locally connected (cf. [10], p. 273). The change of connectedness to arcwise connectedness creates a new condition which seems to be more restrictive, at least for continua that contain sufficiently many arcs, e.g., for locally connected ones. In the present paper, a class of continua characterized by this condition is determined (see 2.4). Some related topics are also discussed.

1. Preliminaries. An infinite sequence of subsets of a metric space is said to be a *null-sequence* provided their diameters converge to zero. We say that a continuum X is *finitely Suslinian* [12] provided each infinite sequence of mutually disjoint subcontinua of X is a null-sequence. All finitely Suslinian continua are hereditarily locally connected (see [12], p. 132).

1.1. *If X is a locally connected continuum which is not hereditarily locally connected, then there exists a number $\varepsilon > 0$ and an infinite sequence of mutually disjoint arcs $A_n \subset X$ such that* (1)

(1) $$\operatorname{diam} A_n > \varepsilon \quad \textit{and} \quad A_n \cap \operatorname*{Ls}_{k\to\infty} A_k = \emptyset \quad (n = 1, 2, \ldots).$$

Proof. Since the continuum X is not hereditarily locally connected, it contains a non-degenerate convergence continuum C (see [10], p. 269). This means there exists an infinite sequence of mutually disjoint continua $C_n \subset X$ such that

(2) $$C = \operatorname*{Lim}_{k\to\infty} C_k \quad \text{and} \quad C_n \cap C = \emptyset \quad (n = 1, 2, \ldots).$$

We write $D_n = \operatorname{cl}(C_{n+1} \cup C_{n+2} \cup \ldots)$ for $n = 1, 2, \ldots$ and observe that

$$D_n = (\operatorname*{Ls}_{i\to\infty} C_{n+i}) \cup (C_{n+1} \cup C_{n+2} \cup \ldots) = C \cup (C_{n+1} \cup C_{n+2} \cup \ldots),$$

(1) We use the symbols Ls, Li and Lim to denote the three kinds of limits of sequences of sets (see [9], p. 335, 337 and 339).

by (2), whence

(3) $$C \subset D_n \quad \text{and} \quad C_n \cap D_n = \emptyset \quad (n = 1, 2, \ldots).$$

A sequence of open subsets $U_n \subset X$ is now defined inductively as follows. Let U_1 be an open set such that $C_1 \subset U_1$, $D_1 \cap \mathrm{cl}\, U_1 = \emptyset$ and U_1 is contained in the 1-neighbourhood of C_1. Assume an integer $n > 1$ is given and the sets U_m are already defined, for $m < n$, such that $D_m \cap \mathrm{cl}\, U_m = \emptyset$. Since $m < n$ implies $C_n \subset D_m$, we have

$$C_n \cap (\mathrm{cl}\, U_1 \cup \ldots \cup \mathrm{cl}\, U_{n-1}) = \emptyset,$$

whence, by (3), there exists an open set U_n such that $C_n \subset U_n$ with

(4) $$(D_n \cup \mathrm{cl}\, U_1 \cup \ldots \cup \mathrm{cl}\, U_{n-1}) \cap \mathrm{cl}\, U_n = \emptyset \quad (n = 2, 3, \ldots)$$

and U_n is contained in the $(1/n)$-neighbourhood of C_n. The last condition implies that

(5) $$\operatorname*{Ls}_{n\to\infty} U_n \subset \operatorname*{Ls}_{n\to\infty} C_n.$$

Let $\varepsilon = \frac{1}{2}\operatorname{diam} C$. The continuum C being non-degenerate, the number ε is positive and, by (2), all but a finite number of the continua C_n have diameters greater than ε. Without loss of generality, we can assume that $\operatorname{diam} C_n > \varepsilon$ for $n = 1, 2, \ldots$ Since $C_n \subset U_n$, a component of U_n, say V_n, contains C_n, and therefore V_n contains a pair of points whose distance exceeds ε. But V_n is a component of an open subset of the locally connected continuum X. Thus V_n is an arcwise connected set (see [10], p. 253 and 257), and we have an arc $A_n \subset V_n$ joining two points distant more than ε apart. Consequently, $A_n \subset U_n$ and

$$\operatorname*{Ls}_{k\to\infty} A_k \subset \operatorname*{Ls}_{k\to\infty} U_k \subset \operatorname*{Ls}_{k\to\infty} C_k = C \subset D_n,$$

by (2), (3) and (5). According to (4), the sets U_n and D_n are disjoint, and so are their subsets A_n and $\operatorname*{Ls}_{k\to\infty} A_k$ $(n = 1, 2, \ldots)$, which completes the proof of (1). Moreover, it follows from (4) that the sets U_n are mutually disjoint. Hence the arcs A_n $(n = 1, 2, \ldots)$ also are mutually disjoint.

1.2. THEOREM. *A locally connected continuum X is finitely Suslinian if and only if each infinite sequence of mutually disjoint arcs contained in X is a null-sequence.*

Proof. The condition stated in 1.2 is, obviously, necessary in order that X be finitely Suslinian. To see that it is sufficient, let us assume that X is a locally connected continuum in which sequences of mutually disjoint arcs are null-sequences. Then X is hereditarily locally connected, by 1.1. If X were not finitely Suslinian, there would exist a number $\varepsilon > 0$ and an infinite sequence of mutually disjoint subcontinua C_n of X such that $\operatorname{diam} C_n > \varepsilon$ for $n = 1, 2, \ldots$ But then C_n would be locally connected, hence also arcwise connected, and we would have mutually disjoint arcs $A_n \subset C_n$ with $\operatorname{diam} A_n > \varepsilon$ for $n = 1, 2, \ldots$, a contradiction. This completes the proof of 1.2.

Remarks. The assumption of local connectedness of X cannot be omitted in 1.2. Indeed, any non-degenerate continuum which contains no arc, e.g., a pseudo-arc, trivially satisfies the condition from 1.2 without being finitely Suslinian. Also, an analogue of Theorem 1.2 for Suslinian continua [12] does not hold. To see this, one can consider a pseudo-arc P and embed it into a locally connected continuum X such that $X \setminus P$ is a subset of the union of a countable collection of arcs (cf. [3], p. 190). Then each collection of mutually disjoint arcs contained in X is countable and, at the same time, the locally connected continuum X is not Suslinian.

1.3. *Suppose that Y is a metric space, $F_n \subset Y$ $(n = 0, 1, \ldots)$ are mutually separated subsets of Y, and $U \subset Y$ is an open subset such that $U \subset F_0 \cup F_1 \cup \ldots$ Then Y is not locally arcwise connected at any point of the set $U \cap F_0 \cap \operatorname{cl}(F_1 \cup F_2 \cup \ldots)$.*

Proof. Let $p \in U \cap F_0 \cap \operatorname{cl}(F_1 \cup F_2 \cup \ldots)$. We prove that each neighbourhood V of p contains a point which cannot be joined with p by means of an arc contained in U. Since p belongs to the closure of $F_1 \cup F_2 \cup \ldots$, we do, in fact, have a point $q \in V \cap F_m$, where $m > 0$. If an arc $A \subset U$ existed such that $p, q \in A$, we would have

$$A = (A \cap F_0) \cup (A \cap F_1) \cup \ldots,$$

according to the assumption that $U \subset F_0 \cup F_1 \cup \ldots$ The sets $A \cap F_n$ $(n = 0, 1, \ldots)$ would be mutually separated and at least two of them, namely, $A \cap F_0$ and $A \cap F_m$, would be non-empty, which is impossible (see [10], p. 173).

1.4. *Suppose X is a metric space, $A_n \subset X$ $(n = 0, 1, \ldots)$ are arcs, $B \subset X$ is an arcwise connected set, and $G \subset X$ is an open subset such that*

(i) $A_n \cap B \neq \emptyset$ *for* $n = 0, 1, \ldots$,

(ii) $A_m \cap A_n \cap G = \emptyset$ *for* $m \neq n$; $m, n = 0, 1, \ldots$,

(iii) $B \cap G = \emptyset$.

Then the union $B \cup (A_0 \cup A_1 \cup \ldots)$ is an arcwise connected set which is not locally arcwise connected at any point of the set $G \cap A_0 \cap \operatorname{cl}(A_1 \cup A_2 \cup \ldots)$.

Proof. Write $Y = B \cup (A_0 \cup A_1 \cup \ldots)$, $F_n = G \cap A_n$ $(n = 0, 1, \ldots)$, and $U = G \cap Y$. By (i), the set Y is arcwise connected and, by (ii), the sets F_n are mutually separated. By (iii), we have $U = F_0 \cup F_1 \cup \ldots$ and also

$$G \cap \operatorname{cl}(A_1 \cup A_2 \cup \ldots) \subset \operatorname{cl}[G \cap (A_1 \cup A_2 \cup \ldots)] = \operatorname{cl}(F_1 \cup F_2 \cup \ldots),$$

whence, by the inclusion $A_0 \subset Y$, we obtain

$$\begin{aligned} G\cap A_0\cap \mathrm{cl}(A_1\cup A_2\cup \ldots) &= G\cap Y\cap G\cap A_0\cap G\cap \mathrm{cl}(A_1\cup A_2\cup \ldots) \\ &\subset U\cap F_0\cap Y\cap \mathrm{cl}(F_1\cup F_2\cup \ldots) \\ &= U\cap F_0\cap \mathrm{cl}_Y(F_1\cup F_2\cup \ldots), \end{aligned}$$

and thus 1.4 follows from 1.3.

1.5. *Suppose that X is a locally connected continuum and $A_n \subset X$ $(n = 0, 1, \ldots)$ are mutually disjoint arcs such that the set $A_0\cap \operatorname{Li}_{k\to\infty} A_k$ is non-degenerate. Then X contains an arcwise connected set which is not locally arcwise connected.*

Proof. Let

$$p,\ q\in A_0\cap \operatorname{Li}_{k\to\infty} A_k, \quad \text{where } p \neq q.$$

Consequently, there exist points $p_k, q_k\in A_k$ $(k = 1, 2, \ldots)$ such that the sequences $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ converge to p and q, respectively. Since X is a locally connected continuum, there exists a locally connected subcontinuum B of X such that $q\in \mathrm{Int} B$ and $p\notin B$ (see [10], p. 257). Thus B is arcwise connected and, moreover, we have $q_k\in B$ for $k > m$, where m is a positive integer. Setting $G = X\setminus B$, we apply 1.4 to conclude that the union $B\cup(A_0\cup A_{m+1}\cup A_{m+2}\cup\ldots)$ is an arcwise connected set which is not locally arcwise connected at the point p.

2. Subsets of finitely Suslinian continua. We now establish a relationship between local arcwise connectedness and the class of finitely Suslinian continua.

2.1. THEOREM. *Each arcwise connected subset of a finitely Suslinian continuum is locally arcwise connected.*

Proof. Suppose, on the contrary, that there exist an arcwise connected subset B of a finitely Suslinian continuum X and a point $p\in B$ such that B is not locally arcwise connected at p. Then there is a neighbourhood G of p and a sequence of points $p_1, p_2, \ldots$ of B converging to p such that p_n cannot be joined with p by means of an arc contained in $B\cap G$ $(n = 1, 2, \ldots)$. Since B is arcwise connected, there exist arcs $A_n \subset B$ such that A_n joins p and p_n $(n = 1, 2, \ldots)$. The union $Y = A_1\cup A_2\cup\ldots$ is a connected subset of the continuum X which is finitely Suslinian, hence also hereditarily locally connected. Thus Y is a locally connected set, and $p\in Y$. Let U be a connected neighbourhood of p in Y such that $\mathrm{cl}_Y U \subset G$. We observe that Y is an F_σ-subset of X, and so is $\mathrm{cl}_Y U$. The points $p_1, p_2, \ldots$ all belong to Y and converge to p, whence $p_m\in U$ for at least one subscript m. Since $\mathrm{cl}_Y U$ is a connected F_σ-subset of the finitely Suslinian continuum X, $\mathrm{cl}_Y U$ is arcwise connected (see [4] and [5], Theorem 3.2). Consequently, there exists an arc $A \subset \mathrm{cl}_Y U$ joining p

and p_m. But then $A \subset Y \subset B$ and $A \subset G$, whence $A \subset B \cap G$, contradicting the main property of G.

2.2. THEOREM. *If X is a locally connected continuum such that each arcwise connected subset of X is locally arcwise connected, then X is hereditarily locally connected.*

Proof. Suppose, on the contrary, that X is not hereditarily locally connected. By 1.1, there exist a number $\varepsilon > 0$ and an infinite sequence of mutually disjoint arcs $A_n \subset X$ which satisfy condition (1). Since X is compact, the sequence $A_1, A_2, \ldots$ has a subsequence which converges to a limit $L \subset X$. Without loss of generality, we can assume that the sequence $A_1, A_2, \ldots$ itself is this subsequence. Hence

$$(6) \qquad L = \operatorname*{Lim}_{k\to\infty} A_k \quad \text{and} \quad A_n \cap L = \emptyset \quad (n = 1, 2, \ldots),$$

where the second equality is a consequence of that of (1). Also, $\operatorname{diam} A_n > \varepsilon$ for $n = 1, 2, \ldots$, and thus $\operatorname{diam} L \geqslant \varepsilon$, so that L is a non-degenerate set. Let $p, q \in L$ be points and $p \neq q$. Since X is a locally connected continuum, X is arcwise connected. Let $A \subset X$ be an arc joining p and q. We notice that, by (6), both p and q belong to $\operatorname*{Li}_{k\to\infty} A_{n_k}$ for each sequence of positive integers $n_1 < n_2 < \ldots$ If we had $A \cap A_{n_k} = \emptyset$ for each term of such a sequence, it would follow from 1.5 that an arcwise connected subset of X would fail to be locally arcwise connected, contrary to the assumption made in 2.2. Consequently, all but a finite number of the arcs A_n intersect the arc A and, without loss of generality, we can assume that $A \cap A_n \neq \emptyset$ for $n = 1, 2, \ldots$ Now, we distinguish two cases as follows.

Case 1. $\operatorname{diam}(A \cap A_n)$ converge to zero. Let $r_n \in A \cap A_n$ $(n = 1, 2, \ldots)$ and let $r_{n_1}, r_{n_2}, \ldots$ be a subsequence converging to a point $r \in X$, where $n_1 < n_2 < \ldots$ We have $p \neq r$ or $q \neq r$. By symmetry, it can be assumed that $p \neq r$. Then also

$$(7) \qquad \operatorname*{Lim}_{k\to\infty}(A \cap A_{n_k}) = \{r\},$$

and $p \in \operatorname*{Li}_{k\to\infty} A_{n_k}$. There are points $p_k \in A_{n_k}$ $(k = 1, 2, \ldots)$ such that p is the limit of $p_1, p_2, \ldots$ By (7), we have $p_k \in A_{n_k} \setminus A$ for $k > k_0$, where k_0 is a positive integer. Let A'_k, for $k > k_0$, be a subarc of A_{n_k} such that $A \cap A'_k = \emptyset$ and A'_k joins the point p_k and a point r'_k whose distance from at least one point of $A \cap A_{n_k}$ is less than $1/k$. Thus $\lim_{k\to\infty} r'_k = r \in A$, according to (7), and we conclude that both p and r belong to $A \cap \operatorname*{Li}_{k\to\infty} A'_k$. The arcs A, $A'_{k_0+1}, A'_{k_0+2}, \ldots$ are mutually disjoint. By 1.5, the continuum X contains an arcwise connected set which is not locally arcwise connected.

Case 2. $\operatorname{diam}(A \cap A_n)$ do not converge to zero. Then there exist points $a_k, b_k \in A \cap A_{m_k}$ $(k = 1, 2, \ldots)$, where $m_1 < m_2 < \ldots$, such that the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ converge to two distinct points $a \in A$ and $b \in A$, respectively. The points a and b belong to the set

(8) $$L' = \operatorname*{Ls}_{k\to\infty} (A \cap A_{m_k})$$

which is a subset of L, by (6). On the other hand, we have

$$A \cap A_{m_k} \subset A \setminus L \subset A \setminus L' \quad (k = 1, 2, \ldots),$$

by (6). It follows that $L' \subset \mathrm{cl}(A \setminus L')$, whence L' is a zero-dimensional closed subset of the arc A. Consequently, there exists a decomposition $L' = M \cup N$ of L' into disjoint compact sets M and N such that $a \in M$ and $b \in N$. Let $U_j, V_j \subset X$ $(j = 1, 2, \ldots)$ be open subsets of X such that $\mathrm{cl}\, U_j \cap \mathrm{cl}\, V_j = \emptyset$ for $j = 1, 2, \ldots$, with

(9) $$M \subset U_j \subset M(1/j) \quad \text{and} \quad N \subset V_j \subset N(1/j) \quad (j = 1, 2, \ldots),$$

where $M(1/j)$ and $N(1/j)$ denote the $(1/j)$-neighbourhoods, in X, of M and N, respectively. We define inductively some arcs A_j'' and positive integers k_j $(j = 0, 1, \ldots)$ such that $A_j'' \subset A_{m_{k_j}}$ and $k_0 < k_1 < \ldots$ Let $A_0'' = A_{m_1}$ and $k_0 = 1$. Let us assume that $j > 0$ is given and A_{j-1}'', k_{j-1} are defined. We find A_j'', k_j in the following way.

First, we observe that since $a \in M \subset U_j$ and $b \in N \subset V_j$, by (9), there exists an integer $k^* > k_{j-1}$ such that $a_k \in U_j$ and $b_k \in V_j$ for $k > k^*$. We also have

$$\operatorname*{Ls}_{k\to\infty} (A \cap A_{m_k}) = L' = M \cup N \subset U_j \cup V_j,$$

by (8) and (9). Thus there is an integer $k_j > k^*$ satisfying the inclusion

$$A \cap A_{m_{k_j}} \subset U_j \cup V_j,$$

whence $A \cap [A_{m_{k_j}} \setminus (U_j \cup V_j)] = \emptyset$. The arc $A_{m_{k_j}}$ contains the points a_{k_j} and b_{k_j} which belong to the open sets U_j and V_j, respectively. Since the closures of these sets are disjoint, there exists a subarc A_j'' of $A_{m_{k_j}}$ such that

$$A_j'' \subset A_{m_{k_j}} \setminus (U_j \cup V_j)$$

and A_j'' joins a point $u_j \in \mathrm{cl}\, U_j$ and a point $v_j \in \mathrm{cl}\, V_j$. As a result, we have $A \cap A_j'' = \emptyset$.

The arcs A_j'' being defined, we can now complete the proof in Case 2 by selecting an appropriate subsequence. Namely, let $j_1 < j_2 < \ldots$ be an infinite sequence of positive integers such that the sequences $u_{j_1}, u_{j_2}, \ldots$ and $v_{j_1}, v_{j_2}, \ldots$ converge to some points $u \in X$ and $v \in X$, respectively. Since

$$u_{j_i} \in \mathrm{cl}\, U_{j_i} \subset \mathrm{cl}\, M(1/j_i) \quad \text{and} \quad v_{j_i} \in \mathrm{cl}\, V_{j_i} \subset \mathrm{cl}\, N(1/j_i) \quad (i = 1, 2, \ldots),$$

by (9), we obtain $u \in M$ and $v \in N$. The sets M and N are disjoint subsets of $L' \subset A$, whence u and v are two distinct points belonging to $A \cap \operatorname{Li}_{i\to\infty} A''_{j_i}$. The arcs $A, A''_{j_1}, A''_{j_2}, \ldots$ are mutually disjoint. As in Case 1, the continuum X contains, by 1.5, an arcwise connected set which is not locally arcwise connected. This contradicts the assumption of the theorem, and the proof of 2.2 is completed.

2.3. THEOREM. *If X is a hereditarily locally connected continuum such that each arcwise connected subset of X is locally arcwise connected, then X is finitely Suslinian.*

Proof. Suppose, on the contrary, that X is not finitely Suslinian. Then there exist a number $\varepsilon > 0$ and an infinite sequence of mutually disjoint continua $C_n \subset X$ such that $\operatorname{diam} C_n > \varepsilon$ for $n = 1, 2, \ldots$ Since X is hereditarily locally connected, each C_n is locally connected, hence also arcwise connected, and there exist mutually disjoint arcs $A_n \subset C_n$ with $\operatorname{diam} A_n > \varepsilon$ $(n = 1, 2, \ldots)$. The sequence $A_1, A_2, \ldots$ has a subsequence which converges to a limit $C \subset X$. Without loss of generality, we can assume that the sequence $A_1, A_2, \ldots$ itself is this subsequence, that is $C = \operatorname{Lim}_{k\to\infty} A_k$. Thus $\operatorname{diam} C \geqslant \varepsilon$, and C is a non-degenerate continuum (see [10], p. 171). Let $p_1, p_2 \in C$ be points and $p_1 \neq p_2$. Since X is a locally connected continuum, there exist locally connected subcontinua $B_1, B_2 \subset X$ such that $p_i \in \operatorname{Int} B_i$ $(i = 1, 2)$ and $B_1 \cap B_2 = \emptyset$. Consequently, there is a positive integer k^* such that

$$(10) \qquad A_k \cap B_1 \neq \emptyset \neq A_k \cap B_2 \quad (k = k^*+1, k^*+2, \ldots).$$

If $A_k \cap C = \emptyset$ for infinitely many subscripts k, the continuum C would be a non-degenerate convergence continuum in X, which is impossible since X is hereditarily locally connected (ibidem, p. 269). It follows that a positive integer $k_0 > k^*$ exists with $A_{k_0} \cap C \neq \emptyset$. Let $q \in A_{k_0} \cap C$. Since $B_1 \cap B_2 = \emptyset$, we have $q \notin B_1$ or $q \notin B_2$. By symmetry, it can be assumed that $q \notin B_1$. Let $G = X \setminus B_1$. The continuum B_1 is arcwise connected and, by (10), the arcs A_k, where $k \geqslant k_0$, satisfy conditions (i) and (ii) of 1.4 for $B = B_1$. Condition (iii) is also satisfied. Moreover, the point q belongs to the set

$$G \cap A_{k_0} \cap C \subset G \cap A_{k_0} \cap \operatorname{cl}(A_{k_0+1} \cup A_{k_0+2} \cup \ldots),$$

since C is the limit of the arcs A_k. By 1.4, $B_1 \cup (A_{k_0} \cup A_{k_0+1} \cup A_{k_0+2} \cup \ldots)$ is an arcwise connected set which is not locally arcwise connected at q. The proof of 2.3 is complete.

2.4. COROLLARY. *A locally connected continuum X is finitely Suslinian if and only if each arcwise connected subset of X is locally arcwise connected.*

Remarks. The assumption of arcwise connectedness of the subset cannot be omitted in 2.1. Indeed, there exists a regular, hence also finitely

Suslinian, continuum containing a non-degenerate connected subset which contains no arc (see [7], p. 109). Also, local connectedness of the continuum X is essential in 2.2 and 2.4. Any non-degenerate continuum X which contains no arc satisfies, trivially, the condition from 2.2 without being locally connected or finitely Suslinian. On the other hand, the condition of the continuum being finitely Suslinian in 2.1 and 2.4 cannot be replaced by that of being hereditarily locally connected. As a matter of fact, there exists a hereditarily locally connected continuum C in the Euclidean 3-space such that C contains a one-to-one continuous image of the real line which is not locally compact (see [13], p. 322). It can be proved that at least one half-line is then transformed onto a set Y which is not locally arcwise connected (see 2.5 below). Clearly, Y is arcwise connected, and also Y is locally connected in this particular situation since $Y \subset C$. By 2.1, each such a hereditarily locally connected continuum C fails to be finitely Suslinian.

2.5. *If $f\colon R^+ \to Y$ is a one-to-one continuous mapping of the half-line $R^+ = \{t\colon t \geqslant 0\}$ onto a metric space Y such that Y is locally arcwise connected at a point $f(t_0)$, then Y is locally compact at $f(t_0)$.*

Proof. If $\lim\limits_{t\to\infty} f(t) = f(t_0)$, the set $Y = f(R^+)$ is compact. Let us assume that this convergence does not take place. It means there exist an open subset $U \subset Y$ and an infinite sequence of positive numbers $x_1 < x_2 < \ldots$ diverging to $+\infty$ such that $f(t_0) \in U$ and $f(x_m) \notin U$ for $m = 1, 2, \ldots$ Then there is a positive integer m_0 such that $t_0 < x_{m_0}$. The sets

$$F_0 = \{f(t)\colon 0 \leqslant t < x_{m_0}\},$$

$$F_n = \{f(t)\colon x_{m_0+n-1} < t < x_{m_0+n}\} \quad (n = 1, 2, \ldots)$$

are mutually separated subsets of Y, and $U \subset F_0 \cup F_1 \cup \ldots$ Since $f(t_0) \in F_0$, it follows from 1.3 that $f(t_0) \in G$, where

$$G = Y \setminus \mathrm{cl}(F_1 \cup F_2 \cup \ldots).$$

The set $G \cap U$ is a neighbourhood of the point $f(t_0)$ in Y, and $G \cap U \subset F_0 \subset f([0, x_{m_0}])$. But the set $f([0, x_{m_0}])$ is compact. Hence Y is locally compact at $f(t_0)$.

3. Final remarks. Let us call a connected space an *(h.l.c.)-space* provided each connected subset of it is locally connected. Thus, by Wilder's theorem mentioned at the beginning of this paper, a continuum is an (h.l.c.)-space if and only if it is hereditarily locally connected. For planar continua, being an (h.l.c.)-space is equivalent to being finitely Suslinian (see [12], p. 132). Keeping this in mind, we can restate Theorem 2.1 to read that each arcwise connected subset of a compact (h.l.c.)-space embeddable in the plane is locally arcwise connected. The necessity of compactness here could be questioned, and the following problem arises.

Problem. *Is it true that each arcwise connected (h.l.c.)-space embeddable in the plane is locally arcwise connected?* (**P 987**)

A similar question concerning locally connected spaces rather than (h.l.c.)-spaces has been asked [6]. The latter question, however, is answered in the negative (see [15], p. 184-185) ([2]). Namely, there exists a planar set which is locally connected and arcwise connected but not locally arcwise connected. Problem P 987 seems to be related to P 253 (see [8], p. 228-229, and [11], p. 267). A counter-example to P 987 would be an arcwise connected planar (h.l.c.)-space which, by 2.1, does not admit any planar (h.l.c.)-compactification. This property is known to be possessed by some arcwise connected planar (h.l.c.)-spaces (see [1], p. 1-2). The problem of the dimension of (h.l.c.)-spaces has also been raised by Duda [2], and still remains unsolved. Some partial results have been recently obtained by Nishiura and Tymchatyn [14].

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