COLLOQUIUM MATHEMATICUM

VOL. XXV 1972 FASC. 1

SOME TYPES OF LACUNARY FOURIER SERIES

BY

M. BOŻEJKO AND T. PYTLIK (WROCŁAW)

1. Let G be a compact abelian group. For a subset E of its dual Γ we define T_E to be the space of all trigonometric polynomials whose Fourier transform is zero outside E. If $S(E)$ consists of all simple functions on Γ vanishing outside E, then the Fourier transformation $\mathfrak{F}$ is a one-to-one map of T_E onto $S(E)$. For f, g in T_E and ξ, η in $S(E)$ let

$$\langle f, g\rangle = \int_G f(x)\overline{g(x)}\,dx, \qquad \langle \xi, \eta\rangle = \sum_{\gamma\in\Gamma} \xi(\gamma)\overline{\eta(\gamma)};$$

then

$$\langle \mathfrak{F}f, \xi\rangle = \langle f, \mathfrak{F}^{-1}\xi\rangle$$

(note that $\mathfrak{F}^{-1}\xi(x) = \sum_{\gamma\in\Gamma} \xi(\gamma)\gamma(x)$).

A function f on G is called an *E-function* if $\hat{f}(\gamma) = 0$ for all γ not in E. The space L_E^p of all E-functions in L^p ($1 \leqslant p \leqslant \infty$) is a closed linear subspace of L^p; for $p \neq \infty$, T_E is dense in it.

Let $\mathfrak{F}_E$ be the restriction of $\mathfrak{F}$ to T_E and $\mathfrak{F}_E^{-1}$ the restriction of $\mathfrak{F}^{-1}$ to $S(E)$. For $1 \leqslant p, q \leqslant \infty$ we define

$$\|\mathfrak{F}_E\|_{p,q} = \sup\{\|\mathfrak{F}f\|_q\colon f\in T_E, \|f\|_p = 1\},$$

$$\|\mathfrak{F}_E^{-1}\|_{p,q} = \sup\{\|\mathfrak{F}^{-1}\xi\|_q\colon \xi\in S(E), \|\xi\|_p = 1\}.$$

Then

$$\|\mathfrak{F}_E\|_{p,q} = \sup\{|\langle\mathfrak{F}f, \xi\rangle| : f\in T_E, \xi\in S(E), \|f\|_p = 1, \|\xi\|_{q'} = 1\},$$

$$\|\mathfrak{F}_E^{-1}\|_{p,q} = \sup\{|\langle\mathfrak{F}f, \xi\rangle| : f\in T_E, \xi\in S(E), \|f\|_{q',E} = 1, \|\xi\|_p = 1\},$$

where $q' = (q-1)/q$, $1' = \infty$, $\infty' = 1$, and

$$\|f\|_{p,E} = \inf\{\|f+g\|_p\colon g\in T_{\Gamma\setminus E}\}$$

is the quotient norm. Further we have formulas

$$\|\mathfrak{F}_E\|_{p,q} \leqslant \|\mathfrak{F}_E^{-1}\|_{q',p'} \tag{1}$$

and, for $F \subset E$,

$$\|\mathfrak{F}_F\|_{p,q} \leqslant \|\mathfrak{F}_E\|_{p,q}, \qquad \|F_F^{-1}\|_{p,q} \leqslant \|\mathfrak{F}_E^{-1}\|_{p,q}. \tag{2}$$

2. We say that $E \subset \Gamma$ belongs to the class $S_{p,q}$ $(1 \leqslant p, q \leqslant \infty)$ if $\|\mathfrak{F}\ \|_{p,q}$ is finite and to the class $S^*_{p,q}$ if $\|\mathfrak{F}_E^{-1}\|_{q',p'}$ is finite.

Inequality (1) shows that the class $S_{p,q}$ $(1 \leqslant p, q \leqslant \infty)$ is larger than the class $S^*_{p,q}$ and (2) shows that both classes are hereditary. It is clear that E is in $S_{p,q}$ or $S^*_{p,q}$ if and only if it is there for every countable subset of E.

The reader has probably noticed that the family $S_{\infty,1}$ is just the family of Sidon sets. The family of Sidon sets play a special role in this paper.

In describing the properties of classes $S_{p,q}$ and $S^*_{p,q}$ it is convenient to use the following functional characterization of them:

THEOREM 1. *Suppose that* $E \subset \Gamma$ *and* $1 \leqslant p, q \leqslant \infty$, $q \neq \infty$, $(p, q) \neq (\infty, 1)$. *The following assertions are equivalent*:

(i) $E \in S_{p,q}$.

(ii) *The Fourier transform of any E-function from* L^p *belongs to* l^q $(\mathfrak{F}L^p_E \subset l^q(E))$.

(iii) *For every* $\xi \in l^{q'}(E)$ *there is a function* $g \in L^{p'}$ *such that* $\hat{g}(\gamma) = \xi(\gamma)$ *for* $\gamma \in E$ $(l^{q'}(E) \subset \mathfrak{F}L^{p'}|_E)$.

THEOREM 1*. *Suppose that* $E \subset \Gamma$ *and* $1 < p, q < \infty$. *The following assertions are equivalent*:

(i*) $E \in S^*_{p,q}$.

(ii*) *The restriction to E of the Fourier transform of any function from* L^p *is in* $l^q(E)$ $(\mathfrak{F}L^p|_E \subset l^q(E))$.

(iii*) *Any function from* $l^{q'}(E)$ *is the Fourier transform of an E-function from* $L^{p'}$ $(l^{q'}(E) \subset \mathfrak{F}L^{p'}_E)$.

Proof of Theorem 1. The case $p \neq \infty$.

(i) $\Rightarrow$ (ii). Since T_E is everywhere dense in L^p_E, $\mathfrak{F}$ can be uniquely extended to a continuous linear map of L^p_E into $l^q(E)$.

(ii) $\Rightarrow$ (iii). By the closed graph theorem it follows that $\mathfrak{F}$ is a continuous operator on L^p_E into $l^q(E)$; in particular, for ξ in $l^{q'}(E)$ the functional

$$f \to \sum_{\gamma \in E} \hat{f}(\gamma)\overline{\xi(\gamma)}$$

is continuous on L^p_E, and by the Hahn-Banach theorem it can be extended to a continuous functional on L^p. This extension must be of the form

$$\sum_{\gamma \in E} \hat{f}(\gamma)\overline{\xi(\gamma)} = \int_G f(x)\overline{g(x)}\,dx,$$

where $g \in L^{p'}$. Putting $f = \gamma$ $(\gamma \in E)$, we have $\xi(\gamma) = \hat{g}(\gamma)$.

(iii) $\Rightarrow$ (i). Let $\xi \in l^{q'}(E)$ and let $g, g_1 \in L^{p'}$ be such that $\xi(\gamma) = \hat{g}(\gamma) = \hat{g}_1(\gamma)$ for all $\gamma \in E$. Then $g_1 - g \in L^{p'}_{\Gamma\setminus E}$.

Define an operator $A\colon l^{q'}(E) \to L^{p'}/L^{p'}_{\Gamma\setminus E}$ by putting $A\xi = g + L^{p'}_{\Gamma\setminus E}$. Notice that, by the closed graph theorem, A is continuous and that for any $\xi \in S(E)$

$$A\xi = \mathfrak{F}^{-1}\xi + L^{p'}_{\Gamma\setminus E}.$$

Therefore if $f \in T_E$, $\xi \in S(E)$ and $\|f\|_p = \|\xi\|_{q'} = 1$, then

$$|\langle \mathfrak{F}f, \xi\rangle| = |\langle f, g\rangle| = \Big|\int_G f(x)\overline{g(x)}\,dx\Big| \leqslant \|g\|_{p',E} \leqslant \|A\|,$$

where $g = \mathfrak{F}^{-1}\xi$. Thus $E \in S_{p,q}$ and the proof is completed.

For the case $p = \infty$ the method used in 5.7.3 of [3] (for Sidon sets) leads to the result $l^q(E) \subset \mathfrak{F}M|_E$. On account of Theorem 32.46 of [1] asserting that, for $r \in [1, \infty)$ and $l^r(\Gamma) = \mathfrak{F}L^1(G)\cdot l^r(\Gamma)$, every $\xi \in l^{q'}(E)$ is representable as $\hat{f}\cdot\hat{\mu}|_E$ with $f \in L^1(G)$, there is $\xi = (f*\mu)^\wedge|_E$.

Proof of Theorem 1*. (i*) $\Rightarrow$ (ii*). It suffices to prove that if $f \in L^p$, then $\hat{f}|_E$ is a continuous functional on $l^{q'}(E)$ — denote it by F. In fact, if $\xi \in S(E)$, then

$$F(\xi) = \sum_{\gamma\in E}\xi(\gamma)\overline{\hat{f}(\gamma)} = \sum_{\gamma\in E}\xi(\gamma)\overline{\int_G f(x)\overline{\gamma(x)}\,dx}$$
$$= \int_G\Big(\sum_{\gamma\in E}\xi(\gamma)\gamma(x)\Big)\overline{f(x)}\,dx = \int_G(\mathfrak{F}^{-1}\xi)(x)\overline{f(x)}\,dx.$$

Hence

$$|F(\xi)| \leqslant \|\mathfrak{F}^{-1}\xi\|_{p'}\|f\|_p \leqslant B\|f\|_p\|\xi\|_{q'} \qquad (B = \|\mathfrak{F}_E^{-1}\|_{q',p'}).$$

(ii*) $\Rightarrow$ (iii*). By the closed graph theorem, the mapping $f \to \mathfrak{F}f|_E$ is continuous from L^p into $l^q(E)$. Hence for $\xi \in l^{q'}(E)$ the functional $f \to \sum_{\gamma\in\Gamma}\hat{f}(\gamma)\overline{\xi(\gamma)}$ is of the form

$$\sum_{\gamma\in\Gamma}\hat{f}(\gamma)\overline{\xi(\gamma)} = \int_G f(x)\overline{g(x)}\,dx$$

with $g \in L^{p'}$. Putting $f = \gamma \in \Gamma$, we have $\xi(\gamma) = \hat{g}(\gamma)$; in particular, $\hat{g}(\gamma) = 0$ for $\gamma \notin E$.

(iii*) $\Rightarrow$ (i*). It is easy to show that the operator A from $l^{q'}(E)$ into $L^{p'}_E$ defined as the inverse operator of $\mathfrak{F}$ has a closed graph. Hence A is continuous. But for $\xi \in S(E)$ we have $A(\xi) = \mathfrak{F}^{-1}\xi$ and thus

$$\|\mathfrak{F}^{-1}\xi\|_{p'} \leqslant \|A\|\|\xi\|_{q'},$$

which completes the proof of the theorem.

Using an argument similar to that in 5.7.3 of [3] one can prove the following

Remarks. Statements (i)-(iii) in Theorem 1 are equivalent to: in the case $p = 1$

(iv) *The Fourier-Stieltjes transform of any E-measure from $M(G)$ is in $l^q(E)$* ($\mathfrak{F}M_E \subset l^q(E)$);

in the case $p = \infty$

(v) *The Fourier transform of any continuous E-function is in $l^q(E)$* ($\mathfrak{F}C_E \subset l^q(E)$).

3. It is known (see [3], 5.7.7) that if

$$\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| \leqslant B\|f\|_\infty \tag{3}$$

holds for every E-polynomial f on G, then

$$\|f\|_p \leqslant B\sqrt{p}\,\|f\|_2 \quad (2 < p < \infty), \tag{4}$$

$$\|f\|_2 \leqslant 2B\|f\|_1 \tag{5}$$

holds for every E-polynomial f.

These formulas can be written in the following simple form:

$$L_E^p = L_E^2 \quad (2 < p < \infty), \tag{6}$$

$$L_E^2 = L_E^1. \tag{7}$$

Rudin [4] calls a set E to be of type $\Lambda(p)$ if $L_E^p = L_E^1$ $(1 < p < \infty)$, and shows that

E is of type $\Lambda(p)$ if and only if there is an $r\in[1, p)$ such that $L_E^p = L_E^r$.

By (6) and (7) the Sidon sets are of type $\Lambda(p)$ whenever $1 < p < \infty$.

Applying the Hausdorff-Young theorem (asserting that $\mathfrak{F}L_E^p \subset l^{p'}(E)$ for all $E \subset \Gamma$ and $1 \leqslant p < 2$) in the case $1 < p < 2$ and the Parseval equality in the case $p = 2$ we thus obtain $\Lambda(p) \subset S_{1,p'}$ $(1 < p \leqslant 2)$ and $\Lambda(p') = S^*_{p,2}$ $(1 \leqslant q < 2)$.

In general, if q and p are arbitrary, it may happen that classes $S_{p,q}$ and $S^*_{p,q}$ consist of finite sets only. Let us analyse the situation more closely.

If the class $S_{p,q}$ contains an infinite set E, then by the general theorem (cf. [2]) E contains an infinite Sidon subset, and since $S_{p,q}$ is hereditary, we may assume that E is a Sidon set. Then $L_E^2 = L_E^p$ for all $1 \leqslant p < \infty$ and hence

$$l^2(E) = \mathfrak{F}L_E^2 \subset l^q(E).$$

But this is possible only if $q \geqslant 2$.

Thus we see that the class $S_{p,q}$ (and so the class $S^*_{p,q}$) with $1 \leqslant p < \infty$ and $1 \leqslant q < 2$ consists of finite sets only.

A similar situation occurs for classes $S^*_{\infty,q}$ with $1 \leqslant q < 2$ and $S^*_{1,q}$ with $2 \leqslant q < \infty$ (as in Theorem 1* we can show that if $E\in S^*_{\infty,q}$, then $l^{q'}(E) \subset \mathfrak{F}L_E^1$, and if $E\in S^*_{1,q}$, then $l^{q'}(E) \subset \mathfrak{F}L_E^\infty$).

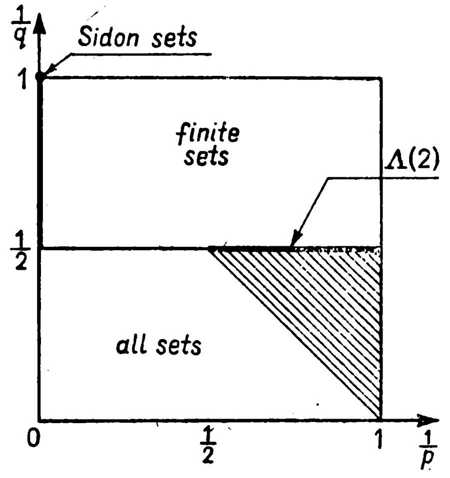
It is evident that

(A) $S_{p_1,q_1} \subset S_{p_2,q_2}$ *and* $S^*_{p_1,q_1} \subset S^*_{p_2,q_2}$, *whenever* $p_1 \leqslant p_2$ *and* $q_1 \leqslant q_2$.

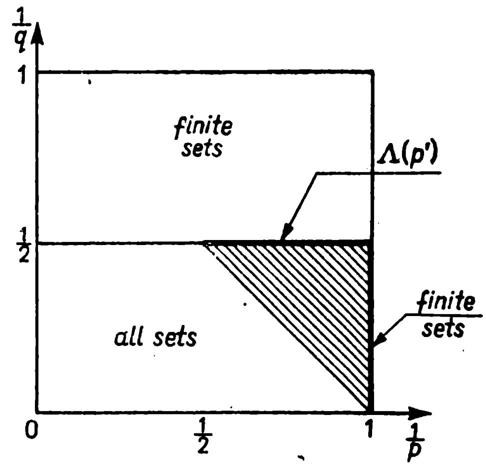
By the Hausdorff-Young Theorem and the formula above, it follows that

(B) *if* $2 \leqslant q \leqslant \infty$ *and* $p \geqslant q'$, *then* $S^*_{p,q} = S_{p,q}$ *are families of all subsets of* Γ.

We illustrate the situation by the diagrams below. Hatched regions represent "new" classes and so does the interval $(\frac{1}{2}, 1)$ of $1/q$-axis. Concerning boundary points, to what of neighbouring domains they belong if it is not denoted in the diagram is to be read off from (A) and (B).



$S_{p,q}$



$S^*_{p,q}$

4. We shall give some additional informations about the classes $S_{p,q}$ and $S^*_{p,q}$.

First we observe that the union of two sets from the class $S^*_{p,q}$ is again in this class. This is an immediate consequence of Theorem 1* (iii*), since it suffices to prove it for disjoint sets. We do not know whether a similar result is true for the classes $S_{p,q}$. **(P 793)**

The Riesz-Thorin interpolation theorem applied to the operator $\mathfrak{F}_E^{-1}$ with domain $S(E)$ shows that

If $0 \leqslant \lambda \leqslant 1$ *and*

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}, \quad \frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2},$$

then

$$S^*_{p,q} \supset S^*_{p_1,q_1} \cap S^*_{p_2,q_2}.$$

Indeed,

$$\frac{1}{p'} = \frac{\lambda}{p_1'} + \frac{1-\lambda}{p_2'} \quad \text{and} \quad \frac{1}{q'} = \frac{\lambda}{q_1'} + \frac{1-\lambda}{q_2'},$$

so that

$$\|\mathfrak{F}_E^{-1}\|_{q',p'} \leqslant \|\mathfrak{F}_E^{-1}\|_{q_1',p_1'}^{\lambda} \|\mathfrak{F}_E^{-1}\|_{q_2',p_2'}^{1-\lambda}$$

for every subset E of Γ, which proves our assertion. Again we do not know whether a similar result is true for the classes $S_{p,q}$. (**P 794**)

A result similar to Theorem 5.7.7 of [3] that Sidon sets are of $\Lambda(p)$-type is the following

THEOREM 2. *If* $1 \leqslant q < 2$ *and* $1/r = 1/q - 1/2$, *then*

(i) $$S_{\infty,q} \subset S_{1,r},$$

(ii) $$S_{\infty,q} \subset \bigcap_{p>1} S^*_{p,r}.$$

Proof. Let $E \in S_{\infty,q}$ and let F be a Sidon set with the same cardinality as E. (We can assume that E is a countable set.) Suppose that $f \in T_E$. For $a > 0$ let us define a function f_a on $G \times G$ by

$$f_a(x, y) = \sum_{\gamma \in E} \hat{f}(\gamma) |\hat{f}(\gamma)|^a \gamma(x) \nu\gamma(y),$$

where ν is a bijection of E onto F.

By Theorem 1 we can find, for any $y \in G$, a function $g_y \in L^1(G)$ such that

$$\hat{g}_y(\gamma) = |f(\gamma)|^a \nu\gamma(y)$$

and

$$\|g_y\|_1 \leqslant B_1 \Big(\sum_{\gamma \in E} |\hat{g}_y(\gamma)|^{q'} \Big)^{1/q'} = B_1 \|\hat{f}\|_{aq'}^a, \qquad (B_1 = \|\mathfrak{F}_E\|_{\infty,q}),$$

but then $f_a(\cdot, y) = f * g_y$. Hence

$$\int_G |f_a(x, y)|\, dx \leqslant \|f\|_1 \|g_y\|_1 \leqslant B_1 \|f\|_1 \|\hat{f}\|_{aq'}^a$$

and so

$$\int_G \int_G |f_a(x, y)|\, dx\, dy \leqslant B_1 \|f\|_1 \|\hat{f}\|_{aq'}^a.$$

On the other hand, $f_a(x, \cdot) \in T_F$ $(x \in G)$. Therefore, by (5),

$$2B \int_G |f_a(x, y)|\, dy \geqslant \|(f_a(x, \cdot))^\wedge\|_2 = \|\hat{f}\|_{2(1+a)}^{1+a},$$

whence

$$2B \int_G \int_G |f_a(x, y)|\, dy\, dx \geqslant \|\hat{f}\|_{2(1+a)}^{1+a}.$$

Applying Fubini's Theorem we thus have

$$\|\hat{f}\|_{2(1+a)}^{1+a} \leqslant 2BB_1 \|f\|_1 \|\hat{f}\|_{aq'}^a.$$

If we choose a so that $2(1+a) = aq'$, then we obtain $\|\hat{f}\|_r \leqslant 2BB_1 \|f\|_1$. Thus $E \in S_{1,r}$ and (i) is true.

To prove (ii) consider the function f_{-a} $(0 < a < 1)$ defined by

$$f_{-a}(x, y) = \sum_{\gamma\in E} \hat{f}(\gamma)\,|\hat{f}(\gamma)|^{-a}\gamma(x)\overline{\nu\gamma(y)}.$$

The equality $\hat{f}(\gamma) = \big(f_{-a}(\cdot, y)\big)^{\wedge}(\gamma)\,\hat{g}_y(\gamma)$ implies $f = f_{-a}(\cdot, y) * g_y$ for all $y \in G$. Therefore

$$\|f\|_{p'}^{p'} \leqslant \|f_{-a}(\cdot, y)\|_{p'}^{p'}\|\mu_y\|_1^{p'} \leqslant \int_G |f_{-a}(x, y)|^{p'} dx \cdot (B_1\|\hat{f}\|_{aq'}^{a})^{p'} \qquad (p > 1).$$

Integrating over G we obtain

$$\|f\|_{p'}^{p'} \leqslant \int_G\int_G |f_{-a}(x, y)|^{p'} dy\,dx \cdot (B_1\|\hat{f}\|_{aq'}^{a})^{p'}. \tag{8}$$

Clearly, we may suppose $p < 2$. Then, by (4) and by the Parseval equality, we have

$$\|f_{-a}(x, \cdot)\|_{p'} \leqslant B\sqrt{p'}\,\|(f_{-a}(x, \cdot))^{\wedge}\|_2 = B\sqrt{p'}\,\|\hat{f}\|_{2(1-a)}^{1-a},$$

so that

$$\int_G\int_G |f_{-a}(x, y)|^{p'} dy\,dx \leqslant (B\sqrt{p'}\,\|\hat{f}\|_{2(1-a)}^{1-a})^{p'}.$$

By virtue of (8), we obtain

$$\|f\|_{p'} \leqslant BB_1\sqrt{p'}\,\|\hat{f}\|_{2(1-a)}^{1-a}\|\hat{f}\|_{aq'}^{a}.$$

If a is such that $2(1-a) = aq'$, then

$$\|f\|_{p'} \leqslant BB_1\sqrt{p'}\,\|\hat{f}\|_{r'} \tag{9}$$

and hence (ii) follows.

5. Now we are going to deal with an arithmetical property of the sets of $S_{p,q}$ and $S^*_{p,q}$ for the group of integers Z.

Given a subset E of Z and a positive integer N, we denote by $a_E(N)$ the largest integer a such that an arithmetic progression of N terms contains a elements of E.

It is shown by Rudin (see [4], theorem 3.5) that if E is of type $\Lambda(p)$ with $p > 2$, then

$$a_E(N) \leqslant (2B)^2 N^{2/p}, \qquad \text{where } B = \|\mathfrak{F}_E^{-1}\|_{2,p}.$$

Precisely the same argument holds for any of the classes $S^*_{p,q}$. One can verify that

5.1. *If $E \in S^*_{p,q}$, then $a_E(N) \leqslant (2BN^{1/p'})^q$ $(N = 1, 2, \ldots, B = \|\mathfrak{F}_E^{-1}\|_{p,q})$.*

This, together with (9) and Theorem 2, implies that

5.2. *If $E \in S_{\infty,q}$, then there is a constant $A = A(E)$ such that*

$$a_E(N) \leqslant A(\log N)^{q/(2-q)} \qquad (N = 3, 4, \ldots). \tag{10}$$

Indeed, by (9) and 5.1, we have $a_E(N) \leqslant (2BB_1\sqrt{p'}N^{1/p'})^r$, where $r = 2q/(2-q)$ and $1 < p < \infty$. Fix N and take $p' = 2\log N$ with $N \geqslant 3$. This choice of p' makes $N^{1/p'} = e^{1/2}$, and (10) follows for $A = (2\sqrt{2}e^{1/2}BB_1)^r$.

(An analogue of this result for Sidon sets may be found in [4], theorem 3.6.)

5.3. *If* $1 \leqslant q < 2$ *and* $1/r = 1/q - 1/2$, *then*

$$S_{\infty,q} \subsetneq \bigcap_{p>1} S^*_{p,r}.$$

Rudin (see [4], theorem 4.11) has shown that if

$$\lim_{N\to\infty} N^{-\varepsilon}\Phi(N) = 0$$

for each $\varepsilon > 0$, then there exists a set E_0 of type $\Lambda(q)$ for every q for which $a_{E_0}(N) > \Phi(N)$ for infinitely many N.

Putting

$$\Phi(N) = (\log N)^{q/(2-q)+1},$$

we infer that there is a set

$$E_0 \in \bigcap_{p>1} \Lambda(p') = \bigcap_{p>1} S^*_{p,2} \subset \bigcap_{p>1} S^*_{p,r},$$

which is not in $S_{\infty,q}$ because of 5.2.

REFERENCES

[1] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, vol. 2, Berlin 1970.

[2] — and H. S. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proceedings of the Cambridge Philosophical Society 62 (1966), p. 399-420.

[3] W. Rudin, *Fourier analysis on groups*, New York 1962.

[4] — *Trigonometric series with gaps*, Journal of Mathematics and Mechanics 9 (1960), p. 203-227.

INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY

Reçu par la Rédaction le 1. 2. 1971