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AN EXAMPLE OF A LOCALLY UNBOUNDED COMPLETE EXTENSION OF THE p-ADIC NUMBER FIELD

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1. In paper [6] (p. 471, Table 1, and p. 472, Table 3) Mutylin raised the question if there exists a complete minimal and not locally bounded extension of the p-adic number field $\mathbf{Q}_p$ (a field topology $\mathcal{T}$ is said to be *locally bounded* provided there exists a bounded neighbourhood A of zero, i.e., for every neighbourhood U of zero there exists a neighbourhood V such that $AV \subset U$). A field topology $\mathcal{T}$ is called *minimal* if it cannot be non-trivially weakened, i.e., if the only topology weaker than $\mathcal{T}$ is trivial. It is well known (see [5] and [7]) that if $(K, \mathcal{T})$ is a topological field endowed with a minimal topology $\mathcal{T}$, then the completion $\hat{K}$ of K in $\mathcal{T}$ is a field.

In this note we give an example of a complete locally unbounded extension of a normed field.

I am indebted to Professor S. Hartman for valuable remarks concerning this paper.

2. Let k denote a non-trivially normed field with a norm $|a|$, and let $L = k(x)$ and $I = k[x]$, where x is transcendental over k. Let $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ denote an infinite sequence of positive real numbers. We take the sets of finite sums

$$U(\varepsilon) = \Big\{f(x) = \sum_{n\geqslant 0} a_n x^n \in I : \mathop{\forall}_{n} |a_n| < \varepsilon_n\Big\},$$

as a base of the neighbourhoods of zero in I. We extend this topology to L by putting $U(\varepsilon, g) = gU(\varepsilon)$, $g \in I$, and taking the sets $V(\varepsilon, g) = U(\varepsilon, g)/\big(1 + U(\varepsilon, g)\big)$ as neighbourhoods of zero in L. Denote this topology by $\mathcal{T}$. It was shown in [3] that if $k = \mathbf{R}$ (with its usual topology), $(L, \mathcal{T})$ is not a locally bounded topological field. This result is true also for any locally bounded field k [2].

Now, let k be a normed field.

THEOREM 1. *$(L, \mathcal{T})$ is a topological field. Moreover, $\mathcal{T}$ is a locally unbounded field topology.*

This theorem can either be deduced from [2] or else proved directly by a slight modification of the original proof of Gould [3].

We prove that the completion $\hat{L}$ of L in $\mathcal{T}$ is a complete locally unbounded topological field. Hence we obtain

THEOREM 2. *For every complete non-trivially normed field k, there exists a complete locally unbounded extension.*

First, we need the following

THEOREM 3. *Suppose that $(K, \mathcal{T})$ is a topological field, and $\hat{K}$ is the completion of K in $\mathcal{T}$. Then either $\hat{K}$ is a topological field or it has divisors of zero. In other words, $\hat{K}$ cannot be a proper integral domain.*

Proof of Theorem 3. If $\mathcal{T}$ is the discrete topology, there is nothing to do since $\hat{K} = K$.

Hence suppose that $\mathcal{T}$ is a proper field topology and that $R = \hat{K}$ is an integral domain. Since R contains K as a topological subring, and R has a unit element, every non-zero element invertible in K remains invertible in R. It follows from [1] (Lemma 3, p. 755) that R contains no proper closed ideals. On the other hand, every principal ideal of R is closed: $aR = \overline{aR}$ holds for every $a \in R$, $a \neq 0$.

In fact, if a is a unit of R, in particular, if $a \in K^{\times}$, it is clear that $a^{-1}R = R$, whence $aR = \overline{aR} = R$. Now, let $a \in R$, $a \neq 0$, be any non-unit. Since K lies densely in R, there is a net $a_\alpha \in K$ with $a_\alpha \to a$. We can suppose, without lose of generality, that $a_\alpha \neq 0$ for all α's. Let V be any symmetric neighbourhood of zero in R. Then $a_\alpha - a \in V$ for sufficiently large α. It follows that $a_\alpha \in a + V$ and, consequently, $R = a_\alpha R \subset (a+V)R \subset aR + VR \subset R$, whence $aR = R - VR = R + VR$. But $R \subset R + VR = aR$ and, finally, $aR = R$ holds for every non-zero $a \in R$. It proves that R is a field. The continuity of division $x \mapsto x^{-1}$ in R, $x \neq 0$, follows from the continuity of division in the dense subgroup $K^{\times} = K \setminus \{0\}$ of the multiplicative complete group $R^{\times} = R \setminus \{0\}$ of R.

Before proving Theorem 2 we insert

LEMMA. *If the completion $\hat{I}$ of I in $\mathcal{T}$-topology has no zero divisors, then $\hat{L}$ has none.*

Proof. Assume $a, b \in \hat{L}$, $a \neq 0$, $b \neq 0$, and $ab = 0$. Since L is dense n $\hat{L}$ and L is the quotient field of I, we have

$$(1) \qquad a = \lim_n \frac{a_n}{a_n'}, \qquad b = \lim_n \frac{b_n}{b_n'},$$

whence

$$\lim_n \frac{a_n b_n}{a_n' b_n'} = 0 \qquad (a_n, a_n', b_n, b_n' \in I).$$

If $f_n/g_n \xrightarrow{n} 0$ $(f_n, g_n \in I)$, then $f_n \xrightarrow{n} 0$. In fact, by the definition of $\mathcal{T}$, for any ε and $g \in I$, there is an $n_0 \in N$ such that

$$\frac{f_n}{g_n} \in \frac{U(\varepsilon, g)}{1+U(\varepsilon, g)} \quad \text{for all } n \geqslant n_0.$$

Fixing g, we have $f_n = gh_n$, $h_n \in U(\varepsilon)$, for every ε and large n, and so $h_n \xrightarrow{n} 0$ in I. By the continuity of multiplication, we infer that $f_n \xrightarrow{n} 0$ in I.

We shall show that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences. We have

$$a - \frac{a_n}{a_n'} \in \frac{U(\varepsilon, g)}{1+U(\varepsilon, g)},$$

whence $aa_n' - a_n = gh$, and $a_n' = 1+gh_1$ with some h_1, $h \in U(\varepsilon)$. Similarly, $aa_m' - a_m = gh'$, and $a_m' = 1+gh_1'$. Hence $a - a_n = g(h-ah_1)$, $a_m - a = g(ah_1' - h')$, and, finally, $a_m - a_n = g(ah_1'' - h'')$, where $h_1'' = h_1 + h_1'$, $h'' = h+h'$, and h'', $h_1'' \in U(\varepsilon)$. So $a_n - a_m \to 0$ for $n, m \to \infty$.

Since $\hat{I}$ is complete, we have (by (1))

$$a_n \xrightarrow{n} \alpha \in \hat{I}, \quad b_n \xrightarrow{n} \beta \in \hat{I} \quad \text{and} \quad \alpha\beta = 0.$$

Then the assumption of the Lemma yields $\alpha = 0$ or $\beta = 0$, so $a = 0$ or $b = 0$. This completes the proof of the Lemma.

Proof of Theorem 2. It is sufficient to show that $\hat{L}$ has no zero divisors and to apply Theorem 3 together with the obvious remark that $\hat{L}$ is locally unbounded since (by Theorem 1) such is L. In view of the Lemma, we have but to prove that $\hat{I}$ has no zero divisors. Let $ab = 0$ in $\hat{I}$, and $b \neq 0$, where $a = \lim\limits_n a_n$, and $b = \lim\limits_n b_n$ $(a_n, b_n \in I)$. If

$$a_n(x) = \sum_{k\geqslant 0} a_k^{(n)} x^k, \quad b_n(x) = \sum_{k\geqslant 0} b_k^{(n)} x^k,$$

$$c_n(x) = a_n(x) b_n(x) = \sum_{k\geqslant 0} c_k^{(n)} x^k, \quad \text{where } c_k^{(n)} = \sum_{r=0}^{k} a_r^{(n)} b_{k-r}^{(n)},$$

then, since $\lim\limits_n a_n b_n = 0$, we have

$$\forall \varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots) \,\exists\, n_0 \forall n \geqslant n_0 \, |c_k^{(n)}| < \varepsilon_k,$$

and so $c_k^{(n)} \xrightarrow{n} 0$ for every $k = 0, 1, 2, \ldots$ All sequences $\{a_k^{(n)}\}$ and $\{b_k^{(n)}\}$ are Cauchy and so convergent in $\hat{I}$.

Let m be the smallest integer for which

$$\lim_n b_m^{(n)} \neq 0$$

(if there were no such m, we would have $\lim\limits_n b_n = b = 0$, contrary to the assumption). Since

$$c_m^{(n)} = a_0^{(n)} b_m^{(n)} + a_1^{(n)} b_{m-1}^{(n)} + \ldots + a_m^{(n)} b_0^{(n)} \xrightarrow{n} 0,$$

we have $\lim\limits_n a_0^{(n)} = 0$. Since

$$c_{m+1}^{(n)} = a_0^{(n)} b_{m+1}^{(n)} + a_1^{(n)} b_m^{(n)} + \ldots + a_{m+1}^{(n)} b_0^{(n)} \xrightarrow{n} 0,$$

there must be $\lim\limits_n a_1^{(n)} = 0$ and so on. Thus,

$$a_k^{(n)} \xrightarrow{n} 0 \text{ for } k = 0, 1, 2, \ldots \quad \text{and} \quad a = \lim_n a_n(x) = 0.$$

It means that $\hat{I}$ has no zero divisors. The proof of Theorem 2 is complete.

COROLLARY. *The Gould topology $\mathcal{T}$ on L fails to be the intersection of the type V topologies of L.*

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