COLLOQUIUM MATHEMATICUM

VOL. XL 1979 FASC. 2

IRREGULAR CONVEX SETS WITH FIXED-POINT PROPERTY FOR NON-EXPANSIVE MAPPINGS

BY

K. GOEBEL AND T. KUCZUMOW (LUBLIN)

Let X be a Banach space with norm $\|\cdot\|$ and let C be a non-empty bounded closed subset of X. The mapping $T: C \to C$ is said to be *non-expansive* if

$$\|Tx-Ty\| \leqslant \|x-y\| \quad \text{for } x, y \in C.$$

We say that C has the *fixed point property* for non-expansive mappings (shortly, f.p.p.) if each non-expansive mapping $T: C \to C$ has at least one fixed point.

It is known that f.p.p. for the set C depends strongly on the "nice" geometrical properties of the space X or on the set C itself. Among bounded closed convex sets having f.p.p. there are, for example, all compact ones, all sets in uniformly convex space X ([1], [3]), all weakly compact sets having normal structure [6], all weakly compact sets in the space X satisfying so-called Opial's condition [7], all weak-star compact subsets of l^1 [4], and all sets in a very special "geometrically bad" but still reflexive space X_J [5].

Our aim here is not to prove any new sufficient conditions for f.p.p. but to construct several examples of very irregular sets still having f.p.p. but not satisfying the above-mentioned conditions. Our sets are not even weak-star compact. We would like to turn attention to some singularities occurring in this field.

Our method is based on the notion of asymptotic center [2]. Let C and X be as above and let $\{x_n\}$ be a bounded sequence of elements of X. Consider a function $r: X \to \langle 0, \infty)$ such that

$$r(y) = \limsup_n \|x_n - y\|.$$

It is a convex function of y depending obviously also on the sequence $\{x_n\}$. However, we skipped this dependence in the notation, which should not lead to any misunderstanding in this paper. The value $r(y)$ is called the *asymptotic radius* of $\{x_n\}$ at y, and the number

$$r(C) = \inf[r(y): y \in C]$$

is the *asymptotic radius* of $\{x_n\}$ with respect to C. The set

$$A(C) = [y \in C: r(y) = r(C)]$$

is called the *asymptotic center* of $\{x_n\}$ in C. Obviously, $A(C)$ is closed and convex but it may be empty.

The connection between f.p.p. and the asymptotic center is given by the following easy lemmas:

LEMMA 1. *If $T: C \to C$ is a non-expansive mapping, then*

$$\inf[\|x - Tx\|: x \in C] = 0.$$

LEMMA 2. *If $T: C \to C$ is non-expansive and $\{x_n\}$ is a sequence of elements of C such that $x_n - Tx_n \to 0$, then $A(C)$ is invariant under T. Especially, if $A(C)$ contains exactly one point, then it is fixed under T.*

The proofs are standard. The next observation is the following

TRIVIAL THEOREM. *If each sequence $\{x_n\}$ of elements of C contains a subsequence whose asymptotic center in C is non-empty and has f.p.p., then C has f.p.p.*

The theorem follows immediately from our lemmas.

We shall construct our examples for $X = l^1$, so denote by $\{e^i\}$ the standard basis $e^i = \{\delta_{ij}\}$, by P_i the natural projection of l^1 on the space spanned by $e^1, e^2, \ldots, e^i$, and let I be the identity. For any set K, $\operatorname{Conv} K$ will denote the closed convex envelope of K.

LEMMA 3. *If $\{x_n\}$ is a sequence in l^1 converging to x in weak-star topology, then for any $y \in l^1$*

$$r(y) = r(x) + \|y - x\|.$$

Proof. Notice first that for any $i = 1, 2, \ldots$

$$r(x) = \limsup_{n\to\infty} \|(I - P_i)(x_n - x)\|,$$

$$\|x - y\| = \lim_{i\to\infty} \lim_{n\to\infty} \|P_i(x_n - y)\|,$$

$$\lim_{i\to\infty} \|(I - P_i)(x - y)\| = 0.$$

Then the thesis follows from the inequality

$$\begin{aligned}\|P_i(x_n - y)\| + \|(I - P_i)(x_n - x)\| - \|(I - P_i)(x - y)\| &\leqslant \|x_n - y\| \\ &\leqslant \|P_i(x_n - y)\| + \|(I - P_i)(x_n - x)\| + \|(I - P_i)(x - y)\|\end{aligned}$$

by passing to infinity first with n and then with i.

LEMMA 4. *Let $\{x_n\}$ be a sequence of elements of l^1 converging to x in weak-star topology. Then*

$$A(C) = \operatorname{Proj} x, \quad \textit{where } \operatorname{Proj} x = [y \in C: \|x - y\| = \operatorname{dist}(x, C)].$$

Lemma 4 is an immediate consequence of Lemma 3. This lemma is also true without assumption of convexity of C. Also it is worth to notice that the set $\operatorname{Proj} x$ may be empty.

Let us construct now a special set C. Take any bounded sequence of non-negative reals $\{a_i\}$ and put $f^i = (1+a_i)e^i$.

Let

$$C = \operatorname{Conv}\{f^i\} = \Big[x = \sum_{i=1}^{\infty}\lambda_i f^i : \lambda_i \geqslant 0,\ \sum_{i=1}^{\infty}\lambda_i = 1\Big].$$

The set C is closed convex but it is not weak-star compact, since the weak-star limit of $\{f^i\}$ is the origin which does not belong to C. The weak-star closure of C is

$$\bar{C} = \Big[x = \sum_{i=1}^{\infty}\mu_i f^i : \mu_i \geqslant 0,\ \sum_{i=1}^{\infty}\mu_i \leqslant 1\Big].$$

For any $x \in \bar{C}$ put

$$\delta_x = 1 - \sum_{i=1}^{\infty}\mu_i.$$

Obviously, δ_x is well defined, since the representation of x as a combination of $\{f^i\}$ is unique. Finally, let $a = \inf a_i$ and $N_0 = [i : a_i = a]$.

Lemma 5. *For any* $x \in \bar{C}$

$$\operatorname{dist}(x, C) = \delta_x(1+a)$$

and

$$\operatorname{Proj} x = \operatorname{Conv}[x + \delta_x f^i : i \in N_0].$$

Proof. We have

$$\begin{aligned}\operatorname{dist}(x, C) &\leqslant \inf[\|x - (x+\delta_x f^i)\| : i = 1, 2, \ldots]\\ &= \inf[(1+a_i)\delta_x : i = 1, 2, \ldots] = \delta_x(1+a).\end{aligned}$$

On the other hand, if

$$y = \sum_{i=1}^{\infty}\lambda_i f^i \in C \quad \text{and} \quad x = \sum_{i=1}^{\infty}\mu_i f^i \in \bar{C},$$

then

$$\|x-y\| = \sum_{i=1}^{\infty}|\lambda_i - \mu_i|(1+a_i) \geqslant \Big|\sum_{i=1}^{\infty}\lambda_i - \sum_{i=1}^{\infty}\mu_i\Big|(1+a) = \delta_x(1+a).$$

It shows also that for $y \in C \setminus \operatorname{Conv}[x+\delta_x f^i : i \in N_0]$

$$\|x-y\| > \delta_x(1+a).$$

Example 1. *The set C described above has f.p.p. if and only if N_0 is non-empty but finite.*

Proof. Take $\{x_n\}$ to be a weak-star convergent sequence of elements of C and let x be its limit. If N_0 is non-empty but finite, then — in view of Lemmas 4 and 5 — $A(C)$ is compact and, since each sequence contains a weak-star convergent subsequence, the assumptions of our Trivial Theorem are fulfilled.

Suppose now that N_0 is empty. Take $\{a_{i_k}\}$ to be a strictly decreasing subsequence of $\{a_i\}$ such that

$$\lim_{k\to\infty} a_{i_k} = a.$$

Let

$$N_k = [i: a_{i_k} \leqslant a_i < a_{i_{k-1}}] \quad \text{for } k = 1, 2, \dots \ (a_{i_0} = +\infty).$$

Define $T: C \to C$ by

$$(*) \qquad T: x = \sum_{i=1}^{\infty} \lambda_i f^i \to \sum_{k=1}^{\infty} \mu_k f^{i_k+1}, \quad \text{where } \mu_k = \sum_{i\in N_k} \lambda_i .$$

It is non-expansive and fixed point free.

Now, let N_0 be infinite and let $N_0 = [i_1, i_2, \dots]$. Put

$$N_k = [i: i_{k-1} < i \leqslant i_k] \quad \text{for } k = 1, 2, \dots \ (i_0 = 0).$$

We see that T defined by $(*)$ is also non-expansive and fixed point free.

Example 2. *The intersection of two sets having f.p.p. may not have f.p.p.*

Proof. Take any $b > 0$ and construct the set C_1 in the way described above putting $a_1 = 0$ and $a_i = b$ for $i = 2, 3, \dots$ Then construct the set C_2 in the same way putting $a_1 = \frac{1}{2}b$ and $a_i = b$ for $i = 2, 3, \dots$ Then C_1 and C_2 have f.p.p. but

$$C_3 = C_1 \cap C_2 = \mathrm{Conv}[(1+b)e^i: i = 2, 3, \dots]$$

fails to have it.

Example 3. *There exists a sequence of sets $\{C_n\}$ such that $C_1 \supset C_2 \supset C_3 \supset \dots$, and C_n has f.p.p. for $n = 1, 3, 5, \dots$ and does not have it for $n = 2, 4, \dots$ Moreover, this sequence may be such that*

$$C_\infty = \bigcap_{n=1}^{\infty} C_n$$

is non-empty and does have or does not have f.p.p. up to our choice.

Proof. Take any bounded increasing sequence $\{b_n\}$ of positive reals and then take the double indexed sequence $\{a_{in}\}$ of positive reals such that

$$\dots < a_{31} < a_{21} < a_{11} < b_1 < \dots < a_{22} < a_{12} < b_2 < \dots$$

Use the sequence $\{a_{in}\}$ to construct the set C in the following way.

Let N denote the set of integers and let $\varphi: N\times N \to N$ be a 1-1 correspondence. Put

$$f^{\varphi(i,n)} = (1+a_{in})e^{\varphi(i,n)}$$

and select any sequence $\{a_{i_n n}\}$. Now put

$$C_{2n-1} = \mathrm{Conv}[f^{\varphi(i,k)}: a_{ik} \geqslant a_{i_n n}],$$
$$C_{2n} = \mathrm{Conv}[f^{\varphi(i,k)}: a_{ik} > b_n].$$

Then the first part of our statement is proved.

To get the second part it is enough to take first a non-empty subset $N_\infty \subset N$ such that $N\setminus N_\infty$ is infinite, then repeat our construction on the basis vectors with indices $i \notin N_\infty$, and put $f^i = (1+b)e^i$ with $b > b_n$ for all n if $i \in N_\infty$. Then

$$\bigcap_{n=1}^{\infty} C_n = \mathrm{Conv}[f^i: i \in N_\infty]$$

has f.p.p. if N_∞ is finite and does not have it if N_∞ is infinite.

The last example is of a little different nature. Let X_1 be an arbitrary uniformly convex Banach space and put $X = X_1 \times l^1$ with the norm

$$\|(x, y)\|_X = \max[\|x\|_{X_1}, \|y\|_{l^1}].$$

Let C_1 be an arbitrary non-empty bounded closed and convex subset of X_1 and let B be a unit ball in l^1. Put $C = C_1\times B$.

Example 4. *C has f.p.p.*

Proof. First notice that each sequence $\{(x_n, y_n)\}$ of elements of C contains a subsequence with $\{y_n\}$ weak-star convergent. Moreover, the asymptotic center of any sequence in a uniformly convex space contains exactly one point [2]. Let then $\{(x_n, y_n)\}$ be such that w*-$\lim y_n = y$. The asymptotic radius of this sequence with respect to C is equal to $r = \max[r_1, r_2]$, where r_1 is the asymptotic radius of $\{x_n\}$ in C_1 and r_2 is the asymptotic radius of $\{y_n\}$ in B. Let $\{x\}$ be the asymptotic center of $\{x_n\}$ in C_1. The asymptotic center $A(C)$ of $\{(x_n, y_n)\}$ in C is equal to $\{(x, y)\}$ if $r_1 = r_2$. However, if $r_1 < r_2$, then

$$A(C) = [z \in C_1: \limsup_n \|x_n - z\|_{X_1} \leqslant r_2]\times \{y\},$$

which has f.p.p. as it is isometric to the closed convex and bounded subset of X_1. On the other hand, if $r_1 > r_2$, then

$$A(C) = \{x\}\times [z \in B: \limsup_n \|y_n - z\|_{l^1} \leqslant r_1]$$
$$= \{x\}\times [z\in B: \|z-y\|_{l^1} \leqslant r_1 - r_2]$$

according to Lemma 3. This set is isometric to the intersection of two balls in l^1 and such an intersection is weak-star compact. So, in view of [4], it has f.p.p. As we see, the assumptions of Trivial Theorem are fulfilled, so C has f.p.p.

Let us finish with the metamathematical statement, not quite clear but in our opinion in some sense true:

For any sufficient condition for f.p.p. there exists a set having f.p.p., which does not satisfy it.

REFERENCES

[1] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proceedings of the National Academy of Sciences of the U.S.A. 54 (1965), p. 1041-1043.

[2] M. Edelstein, *The construction of an asymptotic center with a fixed-point property*, Bulletin of the American Mathematical Society 78 (1972), p. 206-208.

[3] D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Mathematische Nachrichten 30 (1965), p. 251-258.

[4] L. A. Karlovitz, *On nonexpansive mappings*, Proceedings of the American Mathematical Society 55 (1976), p. 321-325.

[5] — *Existence of fixed points of nonexpansive mappings in a space without normal structure*, Pacific Journal of Mathematics 66 (1976), p. 153-159.

[6] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, American Mathematical Monthly 72 (1965), p. 1004-1006.

[7] E. Lami Dozo, *Multivalued nonexpansive mappings and Opial's condition*, Proceedings of the American Mathematical Society 38 (1973), p. 286-292.

M. CURIE-SKŁODOWSKA UNIVERSITY
INSTITUTE OF MATHEMATICS, LUBLIN

Reçu par la Rédaction le 20. 4. 1977