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ABSTRACT GROTHENDIECK RINGS

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Introduction. There are several inequivalent concepts of equivalence of fields with respect to quadratic forms. We will focus our attention on the equivalences related to Witt and Witt–Grothendieck rings of quadratic forms. Two fields F and K are said to be *WR-equivalent* (*GR-equivalent*) if the Witt (Grothendieck) rings $W(F)$ and $W(K)$ ($G(F)$ and $G(K)$) are isomorphic. Although WR-equivalence is relatively easy to handle, the GR-equivalence is not since Grothendieck ring isomorphisms do not – in general – preserve dimensions, forms or hyperbolic planes (see Chapter IV in [6] and also Lemma 3.3 below).

More satisfactory in this respect is the notion of a strong equivalence. We say F and K are *strongly GR-equivalent* (*strongly WR-equivalent*) if there exists a ring isomorphism $G(F) \to G(K)$ ($W(F) \to W(K)$) sending 1-dimensional forms over F onto 1-dimensional forms over K, or, in other words, a ring isomorphism inducing a natural isomorphism of groups of square classes of the two fields (see [6] for a detailed discussion).

Now the fundamental thing is that WR-equivalence coincides with strong WR-equivalence (Harrison–Cordes, see [1]).

As the examples mentioned above suggest, the two GR-equivalences seem to be more apart and only in [7] we have proved that for non-real fields the two GR-equivalences coincide (but they do not coincide – in general – with WR-equivalences; see Example 1.3 below). The case of formally real fields requires a different approach and has been left open. It is the aim of this paper to fill in this lacuna. We prove here that for formally real fields not only GR- and strong GR-equivalences coincide but the GR-equivalences turn out to be identical with WR-equivalences. On examining the matter more closely we have noticed that all the arguments work all right in the more general setup of Marshall's abstract Witt rings ([4]).

Thus in Section 1 we define abstract Grothendieck rings and rephrase all the necessary results of the classical theory to the abstract case. In Section 2 we analyze the relationship between abstract Grothendieck rings, abstract Witt rings and quaternionic structures. The paper culminates in Section 3 where the main results on the relationship between ring isomorphisms and category isomorphisms of abstract Grothendieck rings are proved.

1. Abstract Grothendieck rings. First we recall the concept of an *abstract Witt ring* as introduced in [4]. This is a pair (R, G_R), where R is a commutative ring with 1 and G_R is an elementary 2-group contained in R', the multiplicative group of invertible elements of R. We require that $-1 \in G_R$ and the following axioms are to hold.

W1. R is additively generated by G_R.

W2. R satisfies the Arason–Pfister properties AP(1) and AP(2).

Here AP(k) means: for $a_1, \ldots, a_n \in G_R$,

$$r = a_1 + \ldots + a_n \in I^k, \quad n < 2^k \Rightarrow r = 0$$

where I is the ideal of R generated by all elements $a+b$, $a, b \in G_R$.

W3. If $a_1 + \ldots + a_n = b_1 + \ldots + b_n$, $n \geqslant 3$, $a_i, b_j \in G_R$, then there are $a, b, c_3, \ldots, c_n \in G_R$ such that

$$a_2 + \ldots + a_n = a + c_3 + \ldots + c_n \quad \text{and} \quad a_1 + a = b_1 + b.$$

The motivating example is $R = W(F)$, the Witt ring of anisotropic quadratic forms over a field F of characteristic not 2.

The following axiomatizes the abstract counterpart of the Witt–Grothendieck ring $G(F)$ of a field F.

Definition. An *abstract Grothendieck ring* (AGR, for short) is a pair $(\mathfrak{R}, \mathfrak{G})$, where $\mathfrak{R}$ is a commutative ring with 1 and $\mathfrak{G}$ is an elementary 2-group contained in $\mathfrak{R}'$ such that the following two axioms hold:

G1. If $a_1 + \ldots + a_n = b_1 + \ldots + b_m$, $a_i, b_j \in \mathfrak{G}$, $n \geqslant 1$, $m \geqslant 1$, then $n = m$.

G2. There is an element $e \in \mathfrak{G}$ such that $\mathbf{Z} \cdot (1+e)$ is an ideal in $\mathfrak{R}$ and the quotient $R = \mathfrak{R}/\mathbf{Z} \cdot (1+e)$ is an abstract Witt ring with $G_R = \{a + \mathbf{Z} \cdot (1+e) \colon a \in \mathfrak{G}\}$.

For any field F of characteristic not two, the Grothendieck ring $G(F)$ is an AGR with $\mathfrak{G} = F'/F'^2$.

PROPOSITION 1.1. *$\mathfrak{R}$ is additively generated by $\mathfrak{G}$.*

Proof. For $A \in \mathfrak{R}$, $h(A) = a_1 + \ldots + a_k + \mathbf{Z}(1+e)$, where h is the canonical homomorphism $\mathfrak{R} \to R$, and $a_1, \ldots, a_k \in \mathfrak{G}$ (a consequence of W1). Thus $A = a_1 + \ldots + a_k + z(1+e)$, for some integer z.

Every element $e \in \mathfrak{G}$ such that G2 holds is said to be a *hyperbolic element* of $\mathfrak{G}$ (or, of $\mathfrak{R}$). On the other hand, every element $e \in \mathfrak{G}$ such that $\mathbf{Z} \cdot (1+e)$ is an ideal in $\mathfrak{R}$ is said to be a *universal element* of $\mathfrak{G}$ (or, of $\mathfrak{R}$).

PROPOSITION 1.2. *$e \in \mathfrak{G}$ is a universal element if and only if $1+e = x+xe$ for every $x \in \mathfrak{G}$.*

Proof. If e is universal and $x \in \mathfrak{G}$, then $x \cdot (1+e) = z \cdot (1+e)$ for some

$z \in Z$. By G1, $z = 1$. Conversely, Proposition 1.1 and $1+e = x+xe$ for every $x \in \mathfrak{G}$ imply that $Z \cdot (1+e)$ is the principal ideal generated by $1+e$.

In general, there are many universal elements in $\mathfrak{R}$. In the Grothendieck ring $G(F)$ of a field F, $\langle e \rangle \in G(F)$ is universal if and only if the form $\langle 1, e \rangle$ is universal, and $\langle -1 \rangle$ is a standard hyperbolic element. However, we shall prove (Corollary 3.5) that for any formally real field F the universal and hyperbolic elements of $G(F)$ coincide. More precisely, for any formally real field F and any universal form $\langle 1, e \rangle$ over F, the quotient $G(F)/Z\langle 1, e \rangle$ is isomorphic to the Witt ring $W(F)$. On the other hand, for non-real fields two hyperbolic elements can produce non-isomorphic Witt rings as the following example shows.

Example 1.3. Let F be a field with the property that all binary quadratic forms over F are universal and assume additionally that the level $s(F) = 2$. Consider $\mathfrak{R} = G(F)$. For $e = \langle -1 \rangle$, R is the classical Witt ring $W(F)$, and since $s = 2$, the additive order of $1 \in R$ is 4. Now for $e = \langle 1 \rangle$, the additive order of $1 \in R$ is 2. Hence the two quotients of $\mathfrak{R}$ are not isomorphic. Let us remark that the second quotient of $\mathfrak{R}$ is a Witt ring too. It is isomorphic to the Witt ring of a field K with the group of square classes of the same cardinality as that of F, with $s(K) = 1$ and all binary forms over K universal. Note also that $G(F) = G(K)$. (See [3], Theorem 3.5, p. 44 for the structure results and [1], p. 407, [2], (3.8) and [5], p. 55–56 for the constructions of F and K.)

For $A = a_1 + \ldots + a_n - b_1 - \ldots - b_m$, $a_i, b_j \in \mathfrak{G}$, put $\dim A = n-m$. This defines a ring homomorphism dim: $\mathfrak{R} \to Z$ (it is well defined by G1) split by the unique ring homomorphism $Z \to \mathfrak{R}$. This proves the first part of the following.

PROPOSITION 1.4. (i) $\mathfrak{R} = Z \cdot 1 \oplus \operatorname{Ker} \dim$, *as additive group.*
(ii) Ker dim *is additively generated by the set* $\{1-a \colon a \in \mathfrak{G}\}$.

Proof. (ii) 0-dimensional elements of $\mathfrak{R}$ are $\sum (a_i - b_i)$, with $a_i, b_i \in \mathfrak{G}$, and $a-b = 1-b-(1-a)$.

Recall that for a Witt ring R any ring homomorphism $\tau \colon R \to Z$ is said to be a signature of R. For an AGR we have just defined the dimension homomorphism dim: $\mathfrak{R} \to Z$. A *signature* of $\mathfrak{R}$ will be any ring homomorphism $\sigma \colon \mathfrak{R} \to Z$ different from the dimension homomorphism. For a ring homomorphism $\sigma \colon \mathfrak{R} \to Z$ to be a signature it is necessary and sufficient that there is an element $x \in \mathfrak{G}$ such that $\sigma(x) = -1$. Observe also that $\sigma(e) = -1$ for every universal element $e \in \mathfrak{G}$. Indeed, $e^2 = 1$ implies $\sigma(e) = \pm 1$, and if $\sigma(e) = 1$, then, by Proposition 1.2, universality implies $\sigma(x) = 1$ for every $x \in \mathfrak{G}$, a contradiction.

It follows $\sigma(1+e) = 0$, so that σ factorizes through every $R = \mathfrak{R}/Z(1+e)$, where e is universal, i.e., $\sigma = \tau \circ h$, where $\tau \colon R \to Z$ is a ring homomorphism (a signature if R is a Witt ring, i.e., if e is hyperbolic). The set of all signatures of $\mathfrak{R}$ will be denoted by $X(\mathfrak{R})$.

PROPOSITION 1.5. (i) *$\mathfrak{R} = \boldsymbol{Z}\cdot 1 \oplus \operatorname{Ker}\sigma$, as additive group, for any $\sigma \in X(\mathfrak{R})$.*

(ii) *Let e be any universal element of $\mathfrak{G}$. Then $\operatorname{Ker}\sigma$ is additively generated by the set $\{1+e\} \cup \{1-a\colon a \in \mathfrak{G}$ and $\sigma(a) = 1\}$.*

Proof. (i) σ is split by the unique ring homomorphism $\boldsymbol{Z} \to \mathfrak{R}$.

(ii) Take a typical $A \in \operatorname{Ker}\sigma$, $A = a_1 + \ldots + a_n - b_1 - \ldots - b_m$, $a_i, b_j \in \mathfrak{G}$. Without loss of generality $n \geqslant m$; hence $A = a_1 - b_1 + \ldots + a_m - b_m + a_{m+1} + \ldots + a_n$. Now, playing with the following four identities exactly the same way we did in [6], p. 12–13, one represents A through the generators specified in (ii). The identities are as follows:

(1) $a-b = 1-b-(1-a)$;

(2) $a-b = eb-ea = 1-ea-(1-eb)$;

(3) $a-b = a+eb-(1+e)$;

(4) $a+b = a-eb+1+e$.

(Recall that if $\sigma(a) = -1$, then $\sigma(ea) = 1$.)

We shall need a little deeper result on the additive structure of an AGR.

PROPOSITION 1.6. *For $\sigma \in X(\mathfrak{R})$ put $U = \operatorname{Ker}\dim \cap \operatorname{Ker}\sigma$ and $T = \{1-a\colon a \in \mathfrak{G}$ and $\sigma(a) = 1\}$. Then*

(i) *U is additively generated by T;*

(ii) *$\operatorname{Ker}\dim = \boldsymbol{Z}(1-e) \oplus U$, for any universal element $e \in \mathfrak{G}$;*

(iii) *$\operatorname{Ker}\sigma = \boldsymbol{Z}(1+e) \oplus U$, for any universal element $e \in \mathfrak{G}$.*

Proof. (i) Let (T) be the additive span of T. By Propositions 1.4.(ii) and 1.5.(ii), $T \subset U$, hence also $(T) \subset U$. Suppose now $A \in U$. Then $A \in \operatorname{Ker}\sigma$ and by 1.5.(ii), $A = z(1+e) + \sum \pm(1-a_i)$, where $z \in \boldsymbol{Z}$ and $1-a_i \in T$. Since $A \in \operatorname{Ker}\dim$, $0 = \dim A = 2z$, whence $z = 0$ and $A \in (T)$. Thus $U = (T)$.

(ii) First observe that $\operatorname{Ker}\dim$ is additively generated by $\{1-e\} \cup T$. Indeed, if $1-a \notin T$, then $1-ea \in T$ and $1-a = 1-e-(1-ea)$. By Proposition 1.4 (ii) we are done. This also shows that $\operatorname{Ker}\dim = \boldsymbol{Z}(1-e)+U$. Now if $A \in \operatorname{Ker}\dim$, $A = z(1-e)+u$, $z \in \boldsymbol{Z}$, $u \in U$, then $\sigma(u) = 0$ and $\sigma(A) = 2z$. Thus z is uniquely determined, and so is u. This proves (ii).

(iii) By Proposition 1.5(ii), $\operatorname{Ker}\sigma = \boldsymbol{Z}(1+e)+U$. If $A \in \operatorname{Ker}\sigma$, $A = z(1+e)+u$, $z \in \boldsymbol{Z}$, $u \in U$, then $\dim u = 0$ and $\dim A = 2z$. Thus z is uniquely determined, and so is u. This proves (iii).

Remark. When $\mathfrak{R}$ has no signatures one can also prove that if e is hyperbolic, then $1-e$ generates a direct summand in $\operatorname{Ker}\dim$. The order of $1-e$ is then finite and equal to the level l of the Witt ring $R = \mathfrak{R}/\boldsymbol{Z}(1+e)$. The proof requires some extra work and can be adapted from the classical case discussed in [6], Theorem III.1.

We shall also use the following fact on prime ideals of an AGR.

PROPOSITION 1.7. *The prime ideals* Ker dim *and* Ker σ, *where* $\sigma \in X(\mathfrak{R})$, *are all distinct and are the complete set of minimal prime ideals of* $\mathfrak{R}$.

Proof. Fix a hyperbolic element $e \in \mathfrak{G}$. First observe that Ker dim is the unique minimal prime ideal $\mathfrak{p}$ with the property that $1+e \notin \mathfrak{p}$. For suppose either $\mathfrak{p}$ is a prime ideal contained in Ker dim or $\mathfrak{p}$ is a minimal prime ideal with $1+e \notin \mathfrak{p}$. In either case $(1+e)(1-a) = 0$ implies $1-a \in \mathfrak{p}$, for every $a \in \mathfrak{G}$. Thus $\mathfrak{p}$ contains all the generators of Ker dim (Proposition 1.4 (ii)) and so $\mathfrak{p} =$ Ker dim. It follows that Ker dim is a minimal prime ideal and the unique one with $1+e \notin \mathfrak{p}$. Further, Ker σ is the inverse image of the minimal prime ideal Ker τ of R (cf. [4], Corollary 4.18, p. 82). Hence, if Ker σ, $\sigma \in X(\mathfrak{R})$, are minimal prime ideals, these are all the minimal prime ideals in $\mathfrak{R}$ containing $1+e$ (i.e., containing the kernel of the canonical homomorphism $\mathfrak{R} \to R$).

Suppose $\mathfrak{p} \subset$ Ker σ, where $\mathfrak{p}$ is a prime ideal. By what we have proved above on Ker dim, we can assume $1+e \in \mathfrak{p}$. Take any $a \in \mathfrak{G}$ with $\sigma(a) = 1$. Since $(1+a)(1-a) = 0$ and $1+a \notin$ Ker σ, we have $1-a \in \mathfrak{p}$. Thus $\mathfrak{p}$ contains all the generators of Ker σ determined in Proposition 1.5.(ii), and so $\mathfrak{p}$ = Ker σ. This finishes the proof.

2. AGRs and *Q*-structures. A *Q-structure* is determined by a surjective mapping $q\colon G \times G \to Q$, where G is an elementary 2-group with a distinguished element $e \in G$ and Q is a pointed set with distinguished point $0 \in Q$, satisfying the following axioms:

Q1. $q(a, b) = q(b, a)$;

Q2. $q(a, ea) = 0$;

Q3. $q(a, b) = q(a, c) \Leftrightarrow q(a, bc) = 0$;

Q4. $q(a, b) = q(c, d) \Rightarrow \exists x \in G$ such that $q(a, b) = q(a, x) = q(c, x)$.

Every Witt ring R determines a Q-structure (cf. [4], Proposition 4.2, p. 65) and every Q-structure determines a Witt ring ([4], p. 39) whose associated Q-structure is isomorphic to the given one. Here by a morphism between two Q-structures (G, Q, q, e) and (G', Q', q', e') we mean a group homomorphism $\alpha\colon G \to G'$ such that $\alpha(e) = e'$ and $q(a, b) = 0$ implies $q'(\alpha a, \alpha b) = 0'$. This makes the class of all Q-structures into a category $\mathcal{QS}$. Also the class of Witt rings is made into a category $\mathcal{AWR}$ when we define a morphism between two Witt rings R and S to be a ring homomorphism $\alpha\colon R \to S$ such that $\alpha(G_R) \subset G_S$. $\mathcal{AWR}$ and $\mathcal{QS}$ are naturally equivalent ([4], Theorem 4.5, p. 68).

We define a morphism between two AGRs $\mathfrak{R}$ and $\mathfrak{S}$ to be a ring homomorphism $\alpha\colon \mathfrak{R} \to \mathfrak{S}$ such that $\alpha(\mathfrak{G}(\mathfrak{R})) \subset \mathfrak{G}(\mathfrak{S})$. This makes the class of AGRs into a category $\mathcal{AGR}$. We say $\mathfrak{R}$ and $\mathfrak{S}$ are *isomorphic as* AGRs if $\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic objects of $\mathcal{AGR}$. In terminology of [6, 7], two fields F and K are strongly GR-equivalent if the Grothendieck rings $G(F)$ and $G(K)$ are isomorphic as AGRs.

As in the case of Witt rings, every AGR determines a Q-structure, although not necessarily a unique one. Given $\mathfrak{R}$ we take $G = \mathfrak{G}(\mathfrak{R})$, $Q = Q(\mathfrak{R}) = \{(1-a)(1-b) \in \mathfrak{R}\colon a, b \in \mathfrak{G}(\mathfrak{R})\}$ and define q by $q(a, b) = (1-a) \times \times (1-b)$. The distinguished elements are $0 = q(1, 1)$ in Q and any universal element e in G.

PROPOSITION 2.1. (i) *For any* AGR, $(\mathfrak{G}(\mathfrak{R}), Q(\mathfrak{R}), q, e)$ *defined above is a Q-structure.*

(ii) *Suppose $e \in \mathfrak{G}(\mathfrak{R})$ is a hyperbolic element. Then the Q-structure associated to $\mathfrak{R}$ is isomorphic to the Q-structure associated to the Witt ring $R = \mathfrak{R}/\mathbf{Z}(1+e)$.*

Proof. Here proof of (i) is virtually the same as that of Proposition 4.2 in [4], p. 65, and to prove (ii) observe that the groups $\mathfrak{G}(\mathfrak{R})$ and $G_R = \{a + \mathbf{Z}(1+e)\colon a \in \mathfrak{G}(\mathfrak{R})\}$ are isomorphic via $h(a) = a + \mathbf{Z}(1+e)$, and $h(e) = -1 + \mathbf{Z}(1+e)$. Moreover, if $q_R\colon G_R \times G_R \to Q_R$ is the mapping of the Q-structure associated to R, then the canonical homomorphism $h\colon \mathfrak{R} \to R$ sends $q(a, b)$ into $q_R(h(a), h(b))$. Hence $q(a, b) = 0$ implies $q_R(h(a), h(b)) = 0$, as required.

Since $\mathfrak{R}$ can have non-isomorphic associated Witt rings (see Example 1.3), it can also have non-isomorphic associated Q-structures. But any Q-structure associated to $\mathfrak{R}$, whose distinguished element $e \in \mathfrak{G}(\mathfrak{R})$ is hyperbolic, determines $\mathfrak{R}$ completely. This follows immediately from the following.

PROPOSITION 2.2. *Suppose the distinguished elements $e \in \mathfrak{G}(\mathfrak{R})$ and $e_1 \in \mathfrak{G}(\mathfrak{S})$ of Q-structures associated to $\mathfrak{R}$ and $\mathfrak{S}$ are hyperbolic. Then each Q-structure morphism $\alpha\colon \mathfrak{G}(\mathfrak{R}) \to \mathfrak{G}(\mathfrak{S})$ lifts uniquely to a morphism $\mathfrak{a}\colon \mathfrak{R} \to \mathfrak{S}$. More precisely, let $\alpha'\colon R \to S$ be the unique lifting of α to a morphism of Witt rings and h, h_1 be the canonical homomorphisms $\mathfrak{R} \to R$ and $\mathfrak{S} \to S$. Then there is a unique morphism $\mathfrak{a}\colon \mathfrak{R} \to \mathfrak{S}$ satisfying $h_1 \circ \mathfrak{a} = \alpha' \circ h$.*

Proof. Since $\mathfrak{G}(\mathfrak{R})$ generates $\mathfrak{R}$ additively, if α lifts at all, it lifts uniquely and

$$\mathfrak{a}(a_1 + \ldots + a_n - b_1 - \ldots - b_m) = \alpha a_1 + \ldots + \alpha a_n - \alpha b_1 - \ldots - \alpha b_m, \quad \text{for } a_i, b_j \in \mathfrak{G}(\mathfrak{R}).$$

We only have to check that $\mathfrak{a}$ is well defined, i.e., if

$$A := a_1 + \ldots + a_n - b_1 - \ldots - b_m = 0,$$

then

$$B := \alpha a_1 + \ldots + \alpha a_n - \alpha b_1 - \ldots - \alpha b_m = 0.$$

$A = 0$ implies $\dim A = 0$ and $A \in \mathbf{Z}(1+e)$. Hence also $\dim B = 0$. Now α can be viewed as the morphism of Q-structures associated to R and S and by

Corollary 4.4, [4], p. 68, α lifts to a morphism α': $R \to S$ and

$$\alpha'(A+Z(1+e)) = \alpha'(a_1 + \ldots + a_n + eb_1 + \ldots + eb_m + Z(1+e))$$
$$= \alpha a_1 + \ldots + \alpha a_n + e_1\alpha(b_1) + \ldots + e_1\alpha(b_m) + Z(1+e_1)$$
$$= B + Z(1+e_1).$$

Now $A \in Z(1+e)$ implies $B \in Z(1+e_1)$ and since dim $B = 0$, we conclude $B = 0$ (by G1). This proves the lifting exists. Finally,

$$h_1 \circ \mathfrak{a}(A) = h_1(B) = B + Z(1+e_1) = \alpha' \circ h(A),$$

as required.

COROLLARY 2.3. *If* α: $\mathfrak{G}(\mathfrak{R}) \to \mathfrak{G}(\mathfrak{S})$ *is a Q-structure isomorphism, then* $\mathfrak{a}$: $\mathfrak{R} \to \mathfrak{S}$ *is an isomorphism of* AGRs.

3. Isomorphisms of AGRs. In this section we study the relationship between ring isomorphisms of AGRs and isomorphisms in the category $\mathcal{AGR}$. The main result (Theorem 3.2) establishes that under some mild conditions the two types of morphisms coincide. The key idea is to rectify in some sense a given ring isomorphism $\mathfrak{R} \to \mathfrak{S}$ to assure sending a given universal element of $\mathfrak{R}$ onto a given universal element of $\mathfrak{S}$ (Lemma 3.3). This has also an interesting application in the classical case (Corollary 3.5).

We begin with the following generalization of Theorem IV.3.2 from [6].

PROPOSITION 3.1. *Let* $\mathfrak{R}$ *and* $\mathfrak{S}$ *be two* AGRs *and suppose the associated Witt rings* $R = \mathfrak{R}/Z(1+e)$ *and* $S = \mathfrak{S}/Z(1+e_1)$ *satisfy* AP(3). *Then the following statements are equivalent.*

(i) $\mathfrak{R}$ *and* $\mathfrak{S}$ *are isomorphic as rings and* $\alpha(e) = e_1$ *for one such isomorphism* α.

(ii) R *and* S *are isomorphic as rings.*

(iii) R *and* S *are isomorphic as abstract Witt rings.*

(iv) $\mathfrak{R}$ *and* $\mathfrak{S}$ *are isomorphic as abstract Grothendieck rings and* $\mathfrak{a}(e) = e_1$ *for one such isomorphism* $\mathfrak{a}$.

Proof. (i) $\Rightarrow$ (ii) α sends $Z(1+e)$ onto $Z(1+e_1)$ and induces a ring isomorphism $R \to S$.

(ii) $\Rightarrow$ (iii) is Marshall's Proposition 4.6 in [4], p. 70.

(iii) $\Rightarrow$ (iv) By [4], Corollary 4.4, p. 68, (iii) implies that the Q-structures associated to R and S are isomorphic. By Proposition 2.1, the Q-structures associated to $\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic and so, by Corollary 2.3, there exists an isomorphism $\mathfrak{a}$: $\mathfrak{R} \to \mathfrak{S}$ of AGRs. It remains to prove that $\mathfrak{a}(e) = e_1$. By Proposition 2.2,

$$h_1 \circ \mathfrak{a}(e) = \alpha' \circ h(e) = \alpha'(-1_R) = -1_S = h_1(e_1).$$

It follows that $\alpha(e)$ and e_1 both belong to the same coset of Ker $h_1 = \boldsymbol{Z}(1+e_1)$. A comparison of dimensions gives $\alpha(e) = e_1$. This proves (iv).

(iv) $\Rightarrow$ (i) is trivial.

THEOREM 3.2. *Let $\mathfrak{R}$ and $\mathfrak{S}$ be two abstract Grothendieck rings with the following two properties.*

(a) *Each $\mathfrak{R}$ and $\mathfrak{S}$ has at least one signature.*

(b) *The associated Witt rings R and S satisfy* AP(3).

Then the following statements are equivalent.

(i) *$\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic as rings.*

(ii) *R and S are isomorphic as rings.*

(iii) *R and S are isomorphic as abstract Witt rings.*

(iv) *$\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic as abstract Grothendieck rings.*

On using Proposition 3.1 we need only the following result.

LEMMA 3.3. *Under the assumption* (a) *above, if $\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic as rings and $e \in \mathfrak{G}(\mathfrak{R})$, $e_1 \in \mathfrak{G}(\mathfrak{S})$ are arbitrary universal elements, then there exists a ring isomorphism $\alpha\colon \mathfrak{R} \to \mathfrak{S}$ such that $\alpha(e) = e_1$.*

Proof. Suppose $\beta\colon \mathfrak{R} \to \mathfrak{S}$ is a given ring isomorphism and $\beta(e) \neq e_1$. β maps minimal prime ideals of $\mathfrak{R}$ onto minimal prime ideals of $\mathfrak{S}$. If $\beta(\text{Ker dim}_{\mathfrak{R}}) \neq \text{Ker dim}_{\mathfrak{S}}$, then, by Proposition 1.7, there is a signature σ_1 on $\mathfrak{S}$ such that $\beta(\text{Ker dim}_{\mathfrak{R}}) = \text{Ker } \sigma_1$, and a signature σ on $\mathfrak{R}$ such that $\beta(\text{Ker } \sigma) = \text{Ker dim}_{\mathfrak{S}}$. The other case is when $\beta(\text{Ker dim}_{\mathfrak{R}}) = \text{Ker dim}_{\mathfrak{S}}$. Then again take any signature σ on $\mathfrak{R}$ and let σ_1 be the signature on $\mathfrak{S}$ such that $\beta(\text{Ker } \sigma) = \text{Ker } \sigma_1$. In either case put

$$U = \text{Ker dim}_{\mathfrak{R}} \cap \text{Ker } \sigma \quad \text{and} \quad U_1 = \text{Ker dim}_{\mathfrak{S}} \cap \text{Ker } \sigma_1.$$

These are ideals in $\mathfrak{R}$ and $\mathfrak{S}$, respectively, and by Propositions 1.4 and 1.6,

$$\mathfrak{R} = \boldsymbol{Z} \cdot 1 \oplus \boldsymbol{Z}(1+e) \oplus U,$$
$$\mathfrak{S} = \boldsymbol{Z} \cdot 1 \oplus \boldsymbol{Z}(1+e_1) \oplus U_1.$$

Observe that $\beta(\boldsymbol{Z} \cdot 1) = \boldsymbol{Z} \cdot 1$ and $\beta(U) = U_1$.

We define now an additive isomorphism $\alpha\colon \mathfrak{R} \to \mathfrak{S}$ by putting

$$\alpha(1) = 1, \quad \alpha(1+e) = 1+e_1, \quad \alpha|_U = \beta|_U.$$

It is easy to check that α is, in fact, a ring isomorphism. We claim that

$$\alpha\big((1+e)\cdot u\big) = \alpha(1+e)\cdot\alpha(u), \quad \text{for } u \in U, \tag{3.3.1}$$

$$\alpha\big((1+e)^2\big) = \big(\alpha(1+e)\big)^2. \tag{3.3.2}$$

Recall that $\boldsymbol{Z}(1+e)$ and U are ideals of $\mathfrak{R}$, hence

$$(1+e)\cdot u \in \boldsymbol{Z}(1+e)\cdot U \subset \boldsymbol{Z}(1+e) \cap U = 0,$$

and similarly

$$\alpha(1+e)\cdot\alpha(u) = (1+e_1)\cdot\beta(u) \in Z(1+e_1)\cdot U_1 \subset Z(1+e_1)\cap U_1 = 0.$$

This proves (3.3.1). Further, $(1+e)^2 = 2(1+e)$, whence (3.3.2). Thus α is the desired ring isomorphism.

COROLLARY 3.4. *If $\mathfrak{R}$ has a signature, then for any two universal elements e and e_1 of $\mathfrak{R}$ the associated rings $R = \mathfrak{R}/Z(1+e)$ and $S = \mathfrak{R}/Z(1+e_1)$ are isomorphic. Thus any universal element is hyperbolic and $\mathfrak{R}$ has exactly one associated Witt ring (up to isomorphism).*

Proof. By Lemma 3.3, there is an automorphism of $\mathfrak{R}$ sending e into e_1 and this induces the required isomorphism of R and S.

COROLLARY 3.5. *Let F be a formally real field and $\langle 1, t\rangle$ be any universal binary form over F. Then $Z\langle 1, t\rangle$ is an ideal in the Grothendieck ring $G(F)$ and the quotient $G(F)/Z\langle 1, t\rangle$ is isomorphic to the Witt ring $W(F)$ of the field F.*

Proof. Apply Corollary 3.4 with $\mathfrak{R} = G(F)$, $e = \langle -1\rangle$ and $e_1 = \langle t\rangle$.

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