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A PROBLEM OF HALMOS ON PROJECTIVE BOOLEAN ALGEBRAS

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Halmos, in his paper [2], raised the following question: "The direct product of a finite number of Boolean algebras each of which is the free sum of countable algebras is necessarily projective; does every projective Boolean algebra have this form?" Algebras which are decomposable in this way will be called Halmos algebras. We give a complete characterization of Halmos algebras showing that they form a very narrow class of algebras. Then we give an example of a projective Boolean algebra which is not a Halmos algebra.

In chapter 1 we list some definitions and propositions. Halmos algebras are studied in chapter 2. In chapter 3 we construct the counterexample.

1. Preliminaries. We shall not prove any proposition in this chapter. The reader should consult [1] for 1.4, [3] for 1.3, 1.5, 1.9, 1.11, 1.12 and [4] for 1.6, 1.8; 1.7 and 1.10 are not proved since they are of purely universal algebraic character and almost trivial.

1.1. Each cardinal $\aleph_\tau$ is identified with the set of all ordinals of cardinality less than $\aleph_\tau$. If A is a set, $\bar{\bar{A}}$ is the cardinality of A. ω and $\aleph_0$ both coincide with the set of natural numbers. For every ordinal i, $1+i$ equals $i+1$ if i is finite and equals i if i is infinite.

$f\colon A \to B$ means that f is a function from A to B. We write $\rightarrowtail$ if f is injective and $\twoheadrightarrow$ if f is surjective. If $f\colon A \to B$ and $g\colon B \to C$, then $g \circ f\colon A \to C$ is the composition of f and g. id_A is the identity map on A.

$\langle a, b\rangle$ is the ordered pair with coordinates a and b; similarly, we shall denote ordered n-tuples by $\langle a_1, \ldots, a_n\rangle$, countable sequences by $\langle a_0, a_1, \ldots\rangle$.

1.2. Concerning Boolean algebras, we shall always identify an algebra with its underlying set. The Boolean operations are denoted by $\wedge$, $\vee$, $'$. A *monomorphism* is an injective homomorphism, an *epimorphism* is a surjective homomorphism, an *isomorphism* is both a monomorphism and an epimorphism. $A \cong B$ means that A and B are isomorphic. 2 is the two-element Boolean algebra, 4 the four-element algebra. A Boolean algebra has at least two elements.

1.3. Let, for $1 \leqslant i \leqslant n$, f_i: $A_i \to B_i$ be a homomorphism. Then

$$f_1 \times \ldots \times f_n: A_1 \times \ldots \times A_n \to B_1 \times \ldots \times B_n$$

is the homomorphism defined by

$$f_1 \times \ldots \times f_n(a_1, \ldots, a_n) = \langle f_1(a_1), \ldots, f_n(a_n) \rangle .$$

$f_1 \times \ldots \times f_n$ is injective (surjective) iff, for every i, f_i is injective (surjective).

The following notation will be used: $\underbrace{\langle a, \ldots, a \rangle}_{n}$ for $\langle a_1, \ldots, a_n \rangle$, where $a_i = a$ for $1 \leqslant i \leqslant n$; $\underbrace{A \times \ldots \times A}_{n}$ for $A_1 \times \ldots \times A_n$, where $A_i = A$ for $1 \leqslant i \leqslant n$, etc.

1.4. We shall need a very special case of direct limits. Let, for every $n \in \omega$, A_n be a Boolean algebra and g_n: $A_n \to A_{n+1}$ be a monomorphism. Then there is a Boolean algebra A and, for every $n \in \omega$, a monomorphism i_n: $A_n \to A$ such that $i_{n+1} \circ g_n = i_n$ and $A = \{i_n(x) \mid n \in \omega,\ x \in A_n\}$. A is called the *direct limit* of the family $\langle\langle A_n \mid n \in \omega \rangle, \langle g_n \mid n \in \omega \rangle\rangle$; it is determined up to isomorphism.

1.5. Let I be a non-void set and, for every $i \in I$, let A_i be a Boolean algebra and X_i its dual space. Let A be the dual algebra of the Boolean space $\prod_{i \in I} X_i$; for $i \in I$, let e_i: $A_i \to A$ be the monomorphism dual to the projection map from $\prod_{i \in I} X_i$ to X_i. $\langle A, \langle e_i \mid i \in I \rangle\rangle$ is called the *coproduct* of $\langle A_i \mid i \in I \rangle$ and denoted by $\coprod_{i \in I} A_i$.

1.6. We define F_ω to be the free Boolean algebra on $\aleph_\omega$ free generators, say on $\{v_i \mid i < \aleph_\omega\}$. For every $n \in \omega$, let then F_n be the subalgebra of F_ω generated by $\{v_i \mid i < \aleph_n\}$. Thus F_n is a free Boolean algebra, and $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\omega$ is a chain of free algebras. If, for every $i \in I$, $A_i \cong 4$, $\coprod_{i \in I} A_i$ is the free Boolean algebra on $\overline{\overline{I}}$ generators. Every countable, atomless Boolean algebra is isomorphic to F_0.

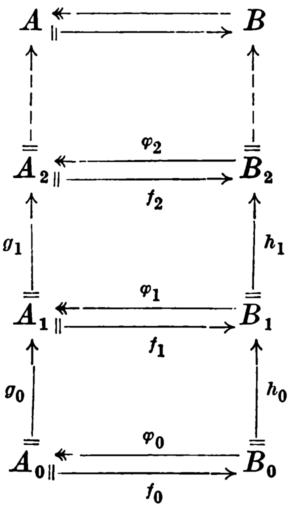
1.7. Let, for every $n \in \omega$, B_n be a free Boolean algebra on $E_n \subseteq B_n$ and let h_n: $B_n \to B_{n+1}$ be a monomorphism such that $\{h_n(x) \mid x \in E_n\} \subseteq E_{n+1}$. Then the direct limit of the family $\langle\langle B_n \mid n \in \omega \rangle, \langle h_n \mid n \in \omega \rangle\rangle$ is free.

1.8. Let A be a Boolean algebra and $a \in A$, $a \neq 0$. Then $\{x \in A \mid x \leqslant a\}$ is, with the partial ordering induced by A, a Boolean algebra. We shall call it the *relative algebra* of A with respect to a and denote it by $A|a$. A is called *homogeneous* if, for every $a \in A$ such that $a \neq 0$, $A \mid a \cong A$.

If A is homogeneous and A non $\cong 2$, $A \times A \cong A$. Every infinite free Boolean algebra is homogeneous.

1.9. A quadruple $\langle A, B, f, \varphi\rangle$ is called a *retraction* if A and B are Boolean algebras, $f\colon A \to B$ and $\varphi\colon B \to A$ are homomorphisms and $\varphi \circ f$ is the identity map on A. A is then said to be a *retract* of B. If, for $1 \leqslant i \leqslant n$, $\langle A_i, B_i, f_i, \varphi_i\rangle$ is a retraction, then $\langle A_1 \times \ldots \times A_n, B_1 \times \ldots \times B_n, f_1 \times \ldots \times f_n, \varphi_1 \times \ldots \times \varphi_n\rangle$ is a retraction.

1.10. Let, for every $n \in \omega$, $\langle A_n, B_n, f_n, \varphi_n\rangle$ be a retraction and $g_n\colon A_n \to A_{n+1}$, $h_n\colon B_n \to B_{n+1}$ be monomorphisms such that $h_n \circ f_n = f_{n+1} \circ g_n$ and $g_n \circ \varphi_n = \varphi_{n+1} \circ h_n$.



Then the direct limit of $\langle\langle A_n \mid n \in \omega\rangle, \langle g_n \mid n \in \omega\rangle\rangle$ is a retract of the direct limit of $\langle\langle B_n \mid n \in \omega\rangle, \langle h_n \mid n \in \omega\rangle\rangle$.

1.11. Let P be a Boolean algebra. P is projective if it satisfies the following condition: if A and B are Boolean algebras, $\pi\colon B \twoheadrightarrow A$ and $f\colon P \to A$ are homomorphisms, then there exists a homomorphism $i\colon P \to B$ such that $f = \pi \circ i$.

1.12. (a) A Boolean algebra is projective iff it is a retract of a free one.

(b) Every countable algebra is projective.

(c) If, for every $i \in I$, A_i is projective, so is $\coprod_{i \in I} A_i$.

(d) If I is finite and, for every $i \in I$, A_i is projective, so is $\prod_{i \in I} A_i$.

1.13. A is said to be a *Halmos algebra* if there exist $n \in \omega$, for every $k \leqslant n$ a Boolean algebra D_k and a set $I_k \neq \emptyset$, and, for every $i \in I_k$, a Boolean algebra B_i such that $2 \leqslant \overline{\overline{B}}_i \leqslant \aleph_0$ and

$$D_k = \coprod_{i \in I_k} B_i, \quad A \cong D_0 \times \ldots \times D_n.$$

By 1.12, every Halmos algebra is projective. We shall construct a projective algebra which is not a Halmos algebra.

2. Structure of Halmos algebras.

2.1. LEMMA. *Let I be a set such that $\bar{\bar{I}} = \aleph_0$ and, for every $i \in I$, let B be a Boolean algebra such that $4 \leqslant \bar{\bar{B}}_i \leqslant \aleph_0$. Then $\coprod_{i\in I} B_i \cong F_0$.*

Proof. Let, for $i \in I$, e_i: $B_i \to \coprod_{i\in I} B_i$ be the canonical map. $\coprod_{i\in I} B_i$, being generated by $\bigcup_{i\in I} \{e_i(x) \mid x \in B_i\}$, is countable.

We shall show that $\coprod_{i\in I} B_i$ is atomless. Adopting the notation of 1.5, we have to show that the algebra of clopen subsets of $\prod_{i\in I} X_i$ is atomless.

Suppose that M is a clopen subset of $\prod_{i\in I} X_i$ such that. $M \neq \emptyset$; we shall construct a clopen subset N of M such that $\emptyset \neq N \neq M$. M is open, so there exist elements $i_0, \ldots, i_n \in I$ and non-void clopen sets $Y_{i_0} \subseteq X_{i_0}, \ldots, Y_{i_n} \subseteq X_{i_n}$ such that

$$\Big\{f \in \prod_{i\in I} X_i | f(i_0) \in Y_{i_0}, \ldots, f(i_n) \in Y_{i_n}\Big\} \subseteq M.$$

Let $i_{n+1} \in I$ s.t. $i_{n+1} \notin \{i_0, \ldots, i_n\}$. $B_{i_{n+1}}$ has at least four elements; so there exists a clopen subset $Y_{i_{n+1}}$ of $X_{i_{n+1}}$ such that $\emptyset \neq Y_{i_{n+1}} \neq X_{i_{n+1}}$. Define

$$N = \Big\{f \in \prod_{i\in I} X_i \mid f(i_0) \in Y_{i_0}, \ldots, f(i_{n+1}) \in Y_{i_{n+1}}\Big\}.$$

Then, $\emptyset \neq N \neq M$ and $N \subseteq M$. Using 1.6, we conclude that $\coprod_{i\in I} B_i \cong F_0$.

2.2. LEMMA. *Let $I \neq \emptyset$ and, for every $i \in I$, let B_i be a countable Boolean algebra. Then $\coprod_{i\in I} B_i$ is countable or free.*

Proof. If, for every $i \in I$, $B_i \cong 2$, $\coprod_{i\in I} B_i$ is isomorphic to 2 and hence countable. Otherwise, there is an $i \in I$ such that $\bar{\bar{B}}_i > 2$. Then

$$\coprod_{i\in I} B_i \cong \coprod \{B_i \mid i \in I, \bar{\bar{B}}_i > 2\}.$$

We may therefore assume that, for every $i \in I$, $\bar{\bar{B}}_i > 2$. If $\bar{\bar{I}} \leqslant \aleph_0$, $\coprod_{i\in I} B_i$ is countable.

Suppose now that $\bar{\bar{I}} > \aleph_0$ and, for every $i \in I$, $\bar{\bar{B}}_i > 2$. Let then $\{I_j | j \in J\}$ be a partition of I such that, for every $j \in J$, $\bar{\bar{I}}_j = \aleph_0$. For every $i \in I$, let C_i be 4. We then have

$$\coprod_{i\in I} B_i \cong \coprod_{j\in J} \Big(\coprod_{i\in I_j} B_i\Big) = \coprod_{j\in J} \Big(\coprod_{i\in I_j} C_i\Big) = \coprod_{i\in I} C_i,$$

by 2.1 and 1.6. This is, again by 1.6, the free algebra on $\bar{\bar{I}}$ generators.

2.3. THEOREM. *A is a Halmos algebra iff*

(a) *A is countable; or*

(b) *A is isomorphic to a product of a finite set of pairwise non-isomorphic, uncountable free algebras; or*

(c) *there is an algebra A_1 satisfying* (a) *and an algebra A_2 satisfying* (b) *such that $A \cong A_1 \times A_2$.*

Proof. It is clear that an algebra satisfying one of conditions (a)-(c) is a Halmos algebra.

Suppose now that $A \cong D_0 \times \ldots \times D_n$, $D_k = \coprod_{i \in I_k} B_i$ for $0 \leqslant k \leqslant n$ and $2 \leqslant \overline{\overline{B}}_i \leqslant \aleph_0$ for $i \in I_k$.

If D_k is countable for every k; A is countable and satisfies (a).

If D_k is uncountable for every k, every D_k is free by 2.2. By 1.8, $D_k \cong D_k \times D_k$; so, after having contracted isomorphic factors, we have a decomposition of A satisfying (b).

If some of the D_k are countable (say $D_{k_0}, \ldots, D_{k_m}$) and some of them are uncountable (say $D_{k_{m+1}}, \ldots, D_{k_n}$), define $A_1 = D_{k_0} \times \ldots \times D_{k_m}$ and $A_2 = D_{k_{m+1}} \times \ldots \times D_{k_n}$. Then A_1 satisfies (a), A_2 satisfies (b), and $A \cong A_1 \times A_2$, so A satisfies (c).

2.4. THEOREM. *Let A be an atomless Halmos algebra. Then A is isomorphic to a product of a finite set of pairwise non-isomorphic infinite free algebras. There are, up to isomorphism, only finitely many relative algebras of A.*

Proof. Since a product is atomless if and only if each of its factors is atomless, we conclude that in case (a) and (c) of Theorem 2.3, the countable factor must be atomless and hence it is isomorphic to F_0. Let now $A = D_0 \times \ldots \times D_n$, where $D_0, \ldots, D_n$ are pairwise non-isomorphic infinite free algebras. Let a be an element of A, say $a = \langle a_0, \ldots, a_n \rangle$, where $\langle a_0, \ldots, a_n \rangle \neq \langle 0, \ldots, 0 \rangle$. Now,

$$A \mid a = D_0 \mid a_0 \times \ldots \times D_n \mid a_n \cong \prod \{D_k \mid a_k \mid a_k \neq 0\} \cong \prod \{D_k \mid a_k \neq 0\},$$

since each of the D_k is homogeneous. Thus, the isomorphism type of $A \mid a$ only depends on the set $\{k \mid 0 \leqslant k \leqslant n,\ a_k \neq 0\}$. There are exactly $2^{n+1}-1$ isomorphism types for the relative algebras of A.

3. The counterexample. Our aim is to construct a diagram in the category of Boolean algebras of the type described in section 1.10. The B_n will be free and the construction of the monomorphisms $h_n\colon B_n \to B_{n+1}$ will guarantee that the direct limit of the B_n is also free. By sections 1.10 and 1.12, the direct limit of the A_n will then be projective. The most difficult step will be the description of the direct limit A of the A_n and the canonical embeddings $i_n\colon A_n \to A$. It is then easily seen, using 2.4, that A is not a Halmos algebra.

We shall first define some special homomorphisms which will help to construct the diagram homomorphisms. Let us recall that F_n is the Boolean algebra freely generated by $\{v_i \mid i < \aleph_n\}$.

3.1. Definition. Let $n, m \in \omega$, $n \leqslant m$. Then $i_{n,m}\colon F_n \to F_m$ and $p_{n,m}\colon F_m \to F_n$ are the homomorphisms defined by

$$i_{n,m}(v_i) = v_i \quad \text{for } i < \aleph_n,$$

$$p_{n,m}(v_i) = \begin{cases} v_i & \text{for } i < \aleph_n, \\ v_0 & \text{for } \aleph_n \leqslant i < \aleph_m, \end{cases}$$

$$F_n \underset{i_{n,m}}{\overset{p_{n,m}}{\leftrightarrows}} F_m.$$

3.2. Lemma. *For $n, m \in \omega$, $n \leqslant m$, $x \in F_n$, we have $i_{n,m}(x) = x$, $p_{n,m}(x) = x$, and $\langle F_n, F_m, i_{n,m}, p_{n,m}\rangle$ is a retraction. For $n \leqslant m \leqslant k$, $i_{m,k} \circ i_{n,m} = i_{n,k}$ and $p_{n,m} \circ p_{m,k} = p_{n,k}$.*

3.3. Definition. For $n \in \omega$, let

$$a_n\colon F_n \to F_n \times F_n$$

be the homomorphism determined by

$$a_n(v_0) = \langle 1, 0\rangle, \qquad a_n(v_{1+i}) = \langle v_i, v_i\rangle;$$

define $\beta_n = a_n^{-1}$ (thus, β_n is a relation!);

$$\lambda_n\colon F_n \to F_n \quad \text{and} \quad \varrho_n\colon F_n \to F_n$$

by $\lambda_n = pr_1 \circ a_n$, $\varrho_n = pr_2 \circ a_n$, where pr_1 (pr_2) is the projection from $F_n \times F_n$ to the first (second) coordinate.

3.4. Lemma. *a_n is an isomorphism. If $n \leqslant m$, then $a_n \subseteq a_m$, $\lambda_n \subseteq \lambda_m$, and $\varrho_n \subseteq \varrho_m$.*

Proof. Let E be $\{a_n(v_i) \mid i < \aleph_n\}$. One easily checks that E is an independent subset of $F_n \times F_n$. The following elements of $F_n \times F_n$ are generated by E: $\langle 0, 1\rangle$, $\langle v_i, 0\rangle$, $\langle 0, v_i\rangle$, $\langle v_i', 0\rangle$, $\langle 0, v_i'\rangle$ (where $i < \aleph_n$). Using disjunctive normal forms, we see that E generates $F_n \times F_n$. Since a_n is the homomorphic extension of a bijection between two systems of free generators, it is an isomorphism (and so is β_n). The rest of the lemma is trivial.

3.5. Definition. Let $n \in \omega$.

$$\begin{array}{ccc}
A_{n+1} = F_0 \times \ldots \times F_n \times F_{n+1} & \underset{f_{n+1}}{\overset{\varphi_{n+1}}{\leftrightarrows}} & \underbrace{F_{n+1} \times \ldots \times F_{n+1}}_{n+2} = B_{n+1} \\
\Big\uparrow g_n & & \Big\uparrow h_n \\
A_n = F_0 \times \ldots \times F_n & \underset{f_n}{\overset{\varphi_n}{\leftrightarrows}} & \underbrace{F_n \times \ldots \times F_n}_{n+1} = B_n.
\end{array}$$

Then define

$$\begin{aligned}
A_n &= F_0 \times \dots \times F_n, \\
B_n &= \underbrace{F_n \times \dots \times F_n}_{n+1}, \\
f_n &= i_{0,n} \times \dots \times i_{n,n}, \\
g_n &= i_{0,0} \times \dots \times i_{n-1,n-1} \times \big((i_{n,n} \times i_{n,n+1}) \circ a_n\big), \\
h_n &= \underbrace{i_{n,n+1} \times \dots \times i_{n,n+1}}_{n} \times \big((i_{n,n+1} \times i_{n,n+1}) \circ a_n\big), \\
\varphi_n &= p_{0,n} \times \dots \times p_{n,n}.
\end{aligned}$$

Thus f_n is simply the inclusion map; g_n and h_n are the maps "splitting up" the last coordinate by the isomorphism a_n.

3.6. LEMMA. (a) *$\langle A_n, B_n, f_n, \varphi_n\rangle$ is a retraction. g_n and h_n are monomorphisms.*

(b) $h_n \circ f_n = f_{n+1} \circ g_n$.

(c) $g_n \circ \varphi_n = \varphi_{n+1} \circ h_n$.

Proof. (a) For $0 \leqslant j \leqslant n$, $\langle F_j, F_n, i_{j,n}, p_{j,n}\rangle$ is a retraction, and so is $\langle A_n, B_n, f_n, \varphi_n\rangle$ by 1.9. The second assertion is trivial by 1.3.

(b) Let $\langle a_0, \dots, a_n\rangle$ be an element of A_n. Then

$$\begin{aligned}
h_n \circ f_n(a_0, \dots, a_n) &= h_n(a_0, \dots, a_n) \\
&= \langle a_0, \dots, a_{n-1}, \lambda_n(a_n), \varrho_n(a_n)\rangle; \\
f_{n+1} \circ g_n(a_0, \dots, a_n) &= f_{n+1}\big(a_0, \dots, a_{n-1}, \lambda_n(a_n), \varrho_n(a_n)\big) \\
&= \langle a_0, \dots, a_{n-1}, \lambda_n(a_n), \varrho_n(a_n)\rangle.
\end{aligned}$$

(c) Let $\langle b_0, \dots, b_n\rangle$ be an element of B_n. Then

$$\begin{aligned}
g_n \circ \varphi_n(b_0, \dots, b_n) &= g_n\big(p_{0,n}(b_0), \dots, p_{n,n}(b_n)\big) \\
&= g_n\big(p_{0,n}(b_0), \dots, p_{n-1,n}(b_{n-1}), b_n\big) \quad (\text{as } p_{n,n} = \mathrm{id}_{F_n}) \\
&= \langle p_{0,n}(b_0), \dots, p_{n-1,n}(b_{n-1}), \lambda_n(b_n), \varrho_n(b_n)\rangle;
\end{aligned}$$

$$\begin{aligned}
\varphi_{n+1} \circ h_n(b_0, \dots, b_n) &= \varphi_{n+1}\big(b_0, \dots, b_{n-1}, \lambda_n(b_n), \varrho_n(b_n)\big) \\
&= \langle p_{0,n+1}(b_0), \dots, p_{n-1,n+1}(b_{n-1}), p_{n,n+1}\big(\lambda_n(b_n)\big), p_{n+1,n+1}\big(\varrho_n(b_n)\big)\rangle.
\end{aligned}$$

The assertion then follows by using the fact that $b_0, \dots, b_{n-1}, \lambda_n(b_n), \varrho_n(b_n) \in F_n$.

3.7. LEMMA. *The direct limit of $\langle\langle B_n \mid n \in \omega\rangle, \langle h_n \mid n \in \omega\rangle\rangle$ is free.*

Proof. Let, for $n \in \omega$, $d_n\colon F_n \to F_n^{n+1}$ be the function defined by

$$d_n(x) = \langle \lambda_n(x), \lambda_n \circ \varrho_n(x), \dots, \lambda_n \circ \varrho_n^{n-1}(x), \varrho_n^n(x)\rangle.$$

d_n is an isomorphism, since it is obtained by iterated application of the isomorphism a_n. Put

$$E_n = \{d_n(v_i) \mid i < \aleph_n\}.$$

E_n freely generates B_n. It is sufficient, by 1.7, to prove that $\{h_n(x) | x \in E_n\} \subseteq E_{n+1}$. So, let $i < \aleph_n$ and consider $h_n \circ d_n(v_i)$. We have

$$\begin{aligned} h_n \circ d_n(v_i) & \\ &= \langle \lambda_n(v_i), \lambda_n \circ \varrho_n(v_i), \ldots, \lambda_n \circ \varrho_n^{n-1}(v_i), \lambda_n \circ \varrho_n^n(v_i), \varrho_n^{n+1}(v_i) \rangle \\ &= \langle \lambda_{n+1}(v_i), \lambda_{n+1} \circ \varrho_{n+1}(v_i), \ldots, \lambda_{n+1} \circ \varrho_{n+1}^n(v_i), \varrho_{n+1}^{n+1}(v_i) \rangle \\ &= d_{n+1}(v_i) \in E_{n+1}. \end{aligned}$$

We shall prove that the direct limit A of the A_n is a certain subalgebra of $\prod_{n \in \omega} F_n$. For the examination of the monomorphisms $i_n\colon A_n \to A$, the homomorphisms ε_n are needed.

3.8. Definition. For $n \in \omega$, let

$$\varepsilon_n\colon F_n \to \prod_{n \leqslant k < \omega} F_k \quad \text{and} \quad i_n\colon A_n \to \prod_{k \in \omega} F_k$$

be defined by

$$\varepsilon_n(x) = \langle \lambda_n(x), \lambda_n \circ \varrho_n(x), \lambda_n \circ \varrho_n^2(x), \ldots \rangle$$

and

$$i_n = i_{0,0} \times \ldots \times i_{n-1,n-1} \times \varepsilon_n.$$

3.9. LEMMA. *ε_n and i_n are homomorphisms. Further, $i_n = i_{n+1} \circ g_n$.*

Proof. ε_n is a function from F_n to $\prod_{n \leqslant k < \omega} F_k$ whose projections λ_n and $i_{n,n+1} \circ \lambda_n \circ \varrho_n$, $i_{n,n+2} \circ \lambda_n \circ \varrho_n^2, \ldots$ are homomorphisms; hence ε_n and i_n are homomorphisms. If $\langle a_0, \ldots, a_n \rangle \in A_n$, then

$$\begin{aligned} i_{n+1} \circ g_n(a_0, \ldots, a_n) & \\ &= i_{n+1}(a_0, \ldots, a_{n-1}, \lambda_n(a_n), \varrho_n(a_n)) \\ &= \langle a_0, \ldots, a_{n-1}, \lambda_n(a_n), \lambda_n \circ \varrho_n(a_n), \lambda_n \circ \varrho_n^2(a_n), \ldots \rangle \\ &= i_n(a_0, \ldots, a_n). \end{aligned}$$

3.10. LEMMA. *i_n is injective.*

Proof. $i_{0,0}, \ldots, i_{n-1,n-1}$ being injective, it suffices by section 1.3 to prove that ε_n is injective. Suppose not. Then there is an element b of F_n such that $b \neq 0$ but $\varepsilon_n(b) = 0$. We may assume that b is a meet of a finite number of free generators or their complements, i.e. there are finite non-void and pairwise disjoint sets M, N, D and E such that $M \cup N \subseteq \aleph_0$, $D \cup E \subseteq \aleph_n \setminus \aleph_0$ and

$$b = \bigwedge_{\mu \in M} v_\mu \wedge \bigwedge_{\nu \in N} v'_\nu \wedge \bigwedge_{\delta \in D} v_\delta \wedge \bigwedge_{\varepsilon \in E} v'_\varepsilon.$$

(We may suppose $n > 0$, for, by $i_0 = i_1 \circ g_0$, it suffices to prove that $i_1, i_2, \ldots$ are injective.) Put

$$b_1 = \bigwedge_{\mu \in M} v_\mu \wedge \bigwedge_{\nu \in N} v'_\nu \quad \text{and} \quad b_2 = \bigwedge_{\delta \in D} v_\delta \wedge \bigwedge_{\varepsilon \in E} v'_\varepsilon,$$

so $b_1 \neq 0 \neq b_2$.

Let k be a natural number such that $k > l$ for every $l \in M \cup N$. Define the function $c\colon F_n \to F_n^{k+1}$ by

$$c(x) = \langle \lambda_n(x), \lambda_n \circ \varrho_n(x), \ldots, \lambda_n \circ \varrho_n^{k-1}(x), \varrho_n^k(x) \rangle.$$

It is then clear that c is an isomorphism and that

$$\varepsilon_n = \big(i_{n,n} \times i_{n,n+1} \times \ldots \times i_{n,n+k-1} \times (\varepsilon_{n+k} \circ i_{n,n+k})\big) \circ c.$$

Let us recall that $a_n(v_0) = \langle 1, 0 \rangle$, $a_n(0) = \langle 0, 0 \rangle$ and $a_n(v_{i+1}) = \langle v_i, v_i \rangle$ if $i < \aleph_0$, and $a_n(v_i) = \langle v_i, v_i \rangle$ if $i \geqslant \aleph_0$. We thus see: if $l = 0$, then

$$c(v_l) = \langle 1, \underbrace{0, \ldots, 0}_{k} \rangle;$$

if $0 < l < k$, then

$$c(v_l) = \langle v_{l-1}, v_{l-2}, \ldots, v_0, 1, \underbrace{0, \ldots, 0}_{k-l} \rangle;$$

if $\aleph_0 \leqslant l < \aleph_n$, then

$$c(v_l) = \langle \underbrace{v_l, \ldots, v_l}_{k+1} \rangle.$$

Therefore,

$$c(b_2) = \langle \underbrace{b_2, \ldots, b_2}_{k+1} \rangle.$$

Case 1. There is some s such that $0 \leqslant s < k$ and $\lambda_n \circ \varrho_n^s(b_1) \neq 0$. Now,

$$\lambda_n \circ \varrho_n^s(b) = \lambda_n \circ \varrho_n^s(b_1) \wedge \lambda_n \circ \varrho_n^s(b_2) = \lambda_n \circ \varrho_n^s(b_1) \wedge b_2.$$

$\lambda_n \circ \varrho_n^s(b_1)$ is generated by free generators with finite indices, and b_2 by free generators with infinite indices. So $\lambda_n \circ \varrho_n^s(b_1)$ and b_2 are independent non-zero elements of F_n and therefore $\lambda_n \circ \varrho_n^s(b) \neq 0$. But then $\varepsilon_n(b) \neq 0$, contradicting our assumption.

Case 2. $\lambda_n \circ \varrho_n^s(b_1) = 0$ for every s such that $0 \leqslant s < k$. Since c is injective, $c(b_1) \neq \langle 0, \ldots, 0 \rangle$; hence, $\varrho_n^k(b_1) \neq 0$. It follows from the examination of $c(v_l)$ for $l \in M \cup N$ that $\varrho_n^k(b_1) \in \{0, 1\}$; so, $\varrho_n^k(b_1) = 1$ and $\varrho_n^k(b) = \varrho_n^k(b_2) = b_2 \neq 0$. But then again,

$$\begin{aligned}\varepsilon_n(b) &= \Big(\big(i_{n,n} \times \ldots \times i_{n,n+k-1} \times (\varepsilon_{n+k} \circ i_{n,n+k})\big) \circ c\Big)(b) \\ &= \big(i_{n,n} \times \ldots \times i_{n,n+k-1} \times (\varepsilon_{n+k} \circ i_{n,n+k})\big)\big(\lambda_n(b), \lambda_n \circ \varrho_n(b), \ldots, \lambda_n \circ \varrho_n^{k-1}(b), b_2\big) \\ &= \langle \lambda_n(b), \lambda_n \circ \varrho_n(b), \ldots, \lambda_n \circ \varrho_n^{k-1}(b), b_2, b_2, b_2, \ldots \rangle \neq 0.\end{aligned}$$

3.11. THEOREM. *The direct limit of* $\langle \langle A_n \mid n \in \omega \rangle, \langle g_n \mid n \in \omega \rangle \rangle$ *is projective, but not a Halmos algebra.*

Proof. Define

$$A = \{i_n(x) \mid n \in \omega, x \in A_n\}.$$

We proved in sections 3.9 and 3.10 that the conditions of 1.4 are satisfied; so A is the direct limit of $\langle\langle A_n \mid n \in \omega\rangle, \langle g_n \mid n \in \omega\rangle\rangle$. Combining 3.6, 1.10, 3.7 and 1.12, we see that A is projective.

Let a be an element of F_n. Then the sequence

$$\langle\underbrace{0, \ldots, 0}_{n}, a, 0, 0, \ldots\rangle$$

is an element of A, since $\langle\underbrace{0, \ldots, 0}_{n}, a, 0\rangle \in A_{n+1}$ and

$$i_{n+1}(\underbrace{0, \ldots, 0}_{n}, a, 0) = \langle\underbrace{0, \ldots, 0}_{n}, a, 0, 0, \ldots\rangle.$$

The relative algebra determined by $\langle 0, \ldots, 0, 1, 0, 0, \ldots\rangle$ in A is therefore

$$\Big\{x \in \prod_{k \in \omega} F_k \mid pr_k(x) = 0 \text{ for every } k \neq n\Big\},$$

where pr_k is the projection onto F_k. This algebra is isomorphic to F_n. So there is an infinite set of non-isomorphic relative algebras of A. Each A_n is atomless and so is A. By section 2.4, we conclude that A is not a Halmos algebra.

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