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ON VARIETIES OF TOPOLOGICAL GROUPS GENERATED BY SOLVABLE GROUPS

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1. In [2] it was shown that any connected Lie group in a variety of topological groups generated by solvable connected Lie groups is solvable. This result is extended here to locally compact groups. (Note that a variety of groups [9] generated by solvable groups may contain non-solvable groups.)

2. A non-empty class of topological groups (not necessarily Hausdorff) is said to be a *variety of topological groups* if it is closed under the operations of taking subgroups, quotients, arbitrary cartesian products and isomorphic images. The smallest variety containing a class $\mathscr{A}$ of topological groups is said to be the *variety generated by* $\mathscr{A}$, and it is denoted by $\mathbf{V}(\mathscr{A})$ (see [1] and [5]-[8]).

If $\mathscr{A}$ is a class of topological groups, then $\mathbf{S}\mathscr{A}$ denotes the class of topological groups (topologically) isomorphic to subgroups of members of $\mathscr{A}$. Similarly we define the operators $\overline{\mathbf{S}}$, $\overline{\mathbf{Q}}$, $\mathbf{C}$ and $\mathbf{D}$, where they denote closed subgroup, separated quotient, arbitrary cartesian product and finite product, respectively.

We will use the following basic result [1] on generating varieties:

THEOREM. *If $\mathscr{A}$ is a class of topological groups and G is a Hausdorff group in $\mathbf{V}(\mathscr{A})$, then $G \in \mathbf{S}\,\mathbf{C}\,\overline{\mathbf{Q}}\,\overline{\mathbf{S}}\,\mathbf{D}(\mathscr{A})$.*

3. Our first theorem extends the classical result that any connected compact solvable group is abelian.

THEOREM 1. *If $\mathscr{A}$ is a class of solvable Hausdorff groups and G is a connected compact group in $\mathbf{V}(\mathscr{A})$, then G is abelian.*

Proof. By the above-mentioned theorem, $G \in \mathbf{S}\,\mathbf{C}\,\overline{\mathbf{Q}}\,\overline{\mathbf{S}}\,\mathbf{D}(\mathscr{A})$. Since finite products, closed subgroups and separated quotients of solvable topological groups are solvable, this implies that G is a subgroup of a product $\prod_{a \in I} H_a$, where each H_a is solvable. Let $P_a(G)$ be the projection of G into H_a for each $a \in I$. Clearly, each $P_a(G)$ is a connected compact solvable group and therefore (see 29.44 of [3]) it is abelian. Thus G is a subgroup of a product of abelian groups and hence it is abelian.

THEOREM 2. *If $\mathscr{A}$ is a class of connected locally compact solvable groups and G is a connected locally compact group in* $\mathbf{V}(\mathscr{A})$, *then G is solvable.*

Proof. Noting Section 4.6 of [4] we see that G has a compact normal subgroup N such that the quotient group G/N is a Lie group. Further, by Section 4.13 of [4], N is contained in a compact connected subgroup M of G. Since $M \in \mathbf{V}(\mathscr{A})$, Theorem 1 shows that M is abelian. Thus N is abelian. We now prove that G/N is solvable and hence G is solvable.

As a consequence of Section 4.6 of [4], we infer that any connected locally compact group H is a subgroup of a product of Lie groups, each of these Lie groups being a quotient of H. This implies that $\mathbf{V}(\mathscr{A}) = \mathbf{V}(\mathscr{B})$, where $\mathscr{B}$ is a class of connected solvable Lie groups. Since G/N is a connected Lie group in $\mathbf{V}(\mathscr{B})$, Corollary 3.4 of [2] implies that G/N is solvable.

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