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ON SOME FUNCTIONAL EQUATION GENERALIZING CAUCHY'S AND D'ALEMBERT'S FUNCTIONAL EQUATIONS

BY

WOJCIECH CHOJNACKI (WARSZAWA)

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0. Introduction. Let G be a compact group and Γ a locally compact Abelian group. Suppose Γ is a G-space in the sense that there is a map

$$G\times\Gamma\ni(g, x)\to gx\in\Gamma$$

such that $g(hx) = (gh)x$, $ex = x$, and $g(x+y) = gx+gy$ for all $g, h\in G$ and all $x, y\in\Gamma$, e being the neutral element of G. Suppose, moreover, that the action of G is measurable in the sense that the function $G\times\Gamma\ni(g, x)\to gx\in\Gamma$ and all the functions $G\times\Gamma\ni(g, y)\to gx+y\in\Gamma$ $(x\in\Gamma)$ are $(m\times\mu, \mu)$-measurable, where m and μ are Haar measures on G and Γ, respectively.

The aim of this paper is to study μ-measurable, in various senses essentially bounded solutions to the functional equation

$$\int_G f(x+gy)\,dm(g) = f(x)f(y) \quad (x, y\in\Gamma). \tag{0.1}$$

Two particular cases of this equation have already been the object of investigations:

1° Cauchy's functional equation

$$f(x+y) = f(x)f(y) \quad (x, y\in\Gamma)$$

resulting from (0.1) upon taking $G = \mathbf{Z}_1$, with normalized Haar measure, to act trivially on Γ;

2° d'Alembert's functional equation

$$f(x+y)+f(x-y) = 2f(x)f(y) \quad (x, y\in\Gamma)$$

resulting from (0.1) if $G = \mathbf{Z}_2$, with normalized Haar measure, is taken to act on Γ by the rule $0\cdot x = x$, $1\cdot x = -x$ $(x\in\Gamma)$.

In either of these cases an arbitrary group Γ is a G-space for a group G of a particularly simple form. Specific groups Γ can apparently be realized as G-spaces for more complicated groups G. A simple example is furnished by the additive group of complex numbers C which is a Z_k-space under the action

$$(g, x) \to \left[\exp\left(\frac{2\pi i}{k} g\right)\right] x \quad (k \in N),$$

and is also a T-space if the circle group T is taken to act on C by multiplication.

Our subsequent considerations will proceed in the following order. First, we find all μ-measurable μ-essentially bounded and μ-essentially non-zero solutions of the equation in question. Next, we discuss generalized complex-valued solutions in the case where G is finite and Γ is discrete; the identity (0.1) in this case is only assumed to hold almost everywhere with respect to certain translation-invariant ideals of subsets of Γ. Finally, we study some generalized solutions whose values lie in commutative semi-simple Banach algebras. As a result in this last case we obtain a generalization of a theorem of de Bruijn [2] on almost additive functions.

It should be pointed out that none of our results will essentially depend on a particular choice of the norming constant $m(G)$. This is due to the fact that a function f satisfies (0.1) if and only if the function cf $(c > 0)$ satisfies the same identity with m replaced by cm.

1. Solutions in $\mathscr{L}^\infty(\Gamma)$. Let $\mathscr{L}^\infty(\Gamma)$ be the space of all complex μ-measurable μ-essentially bounded functions on Γ, $\hat{\Gamma}$ the dual group of Γ.

Given $\chi \in \hat{\Gamma}$, put

$$f(x) = \int_G (gx, \chi)\, dm(g) \quad (x \in \Gamma). \tag{1.1}$$

Clearly, f is in $\mathscr{L}^\infty(\Gamma)$. We claim that f satisfies (0.1). In fact, if $x, y \in \Gamma$, then

$$\begin{aligned}
\int_G f(x+gy)\, dm(g) &= \int_G \left[\int_G (hx+hgy, \chi)\, dm(h)\right] dm(g) \\
&= \int_G (hx, \chi)\left[\int_G (hgy, \chi)\, dm(g)\right] dm(h) \\
&= \int_G (hx, \chi)\left[\int_G (gy, \chi)\, dm(g)\right] dm(h) \\
&= f(x)f(y).
\end{aligned}$$

Conversely, we have the following

THEOREM 1.1. *Any μ-essentially non-zero function in $\mathscr{L}^\infty(\Gamma)$ satisfying (0.1) can be represented in the form (1.1) for some $\chi \in \hat{\Gamma}$.*

This theorem is a particular case of the following more general result:

THEOREM 1.2. *Let w be a μ-essentially non-zero function in $\mathscr{L}^{\infty}(\Gamma)$, Ω a subset of Γ, and f a complex function on Ω. Suppose that for each $y \in \Omega$ the identity*

$$\int_G w(x+gy)\,dm(g) = f(y)\,w(x)$$

holds for μ-almost all x in Γ. Then there exists $\chi \in \hat{\Gamma}$ such that for every $y \in \Omega$

$$f(y) = \int_G (gy, \chi)\,dm(g).$$

Proof. Let $A(\hat{\Gamma})$ be the space of Fourier transforms of functions in $L^1(\Gamma)$ with the norm $\|\varphi\|_{A(\hat{\Gamma})} = \|u\|_1$, where $\varphi = \hat{u}$; we adopt the following convention as regards the Fourier transform:

$$\hat{u}(\chi) = \int_\Gamma u(x)\overline{(x, \chi)}\,d\mu(x) \quad (\chi \in \hat{\Gamma}).$$

Given $y \in \Omega$, define the function ε_y on $\hat{\Gamma}$ to be

$$\varepsilon_y(\chi) = \int_G (gy, \chi)\,dm(g) \quad (\chi \in \hat{\Gamma}).$$

Note that if $\varphi \in A(\hat{\Gamma})$, then $\varepsilon_y \varphi \in A(\hat{\Gamma})$. In fact, if $\varphi = \hat{u}$, then $\varepsilon_y \varphi$ is the Fourier transform of the function

$$x \to \int_G u(x+gy)\,dm(g) \quad (x \in \Gamma),$$

an element of $L^1(\Gamma)$.

Let $\hat{w}$ be the Fourier transform of w regarded as a pseudomeasure on $\hat{\Gamma}$ (cf. [1]), i.e., $\hat{w}$ is the linear continuous functional on $A(\hat{\Gamma})$ defined by

$$\langle \hat{w}, \varphi \rangle = \int_\Gamma w(-x)\,u(x)\,d\mu(x) \quad (\varphi \in A(\hat{\Gamma}),\ \varphi = \hat{u}).$$

We claim that for each $y \in \Omega$ and each $\varphi \in A(\hat{\Gamma})$

$$[(\varepsilon_y - f(y))\varphi]\,\hat{w} = 0. \tag{1.2}$$

In fact, for each $\psi \in A(\hat{\Gamma})$ we have

$$\begin{aligned}
\langle [(\varepsilon_y - f(y))\varphi]\,\hat{w}, \psi \rangle &= \langle \hat{w}, \varepsilon_y \varphi\psi \rangle - f(y)\langle \hat{w}, \varphi\psi \rangle \\
&= \int_\Gamma w(-x)\Big[\int_G (u*v)(x+gy)\,dm(g)\Big]\,d\mu(x) \\
&\quad - f(y)\int_\Gamma w(-x)(u*v)(x)\,d\mu(x) \\
&= \int_\Gamma (u*v)(-x)\Big[\int_G w(x+gy)\,dm(g)\Big]\,d\mu(x) \\
&\quad - f(y)\int_\Gamma w(x)(u*v)(-x)\,d\mu(x) = 0,
\end{aligned}$$

where $\varphi = \hat{u}$, $\psi = \hat{v}$, and $*$ stands for convolution.

Since w is μ-essentially non-zero, the support of $\hat{w}$ is non-void. Let χ be a point in the support of $\hat{w}$. From (1.2) we infer that for every $y \in \Omega$ and every $\varphi \in A(\hat{\Gamma})$ the function $(\varepsilon_y - f(y))\varphi$ vanishes at χ (cf. [1], Theorem 1.3.1). Consequently, for all $y \in \Omega$ we have $\varepsilon_y(\chi) = f(y)$, which is the desired representation.

2. Generalized complex-valued solutions. We begin by recalling certain concepts and introducing some notation.

An *ideal* of subsets of a set X is a family $\mathfrak{J}$ of subsets of X such that

(i) $\emptyset \in \mathfrak{J}$ and $X \notin \mathfrak{J}$;

(ii) if $A \in \mathfrak{J}$ and $B \subset A$, then $B \in \mathfrak{J}$;

(iii) if $A, B \in \mathfrak{J}$, then $A \cup B \in \mathfrak{J}$.

Here the condition that $X \notin \mathfrak{J}$ is not standard.

Let X be a set and $\mathfrak{J}$ an ideal of subsets of X. If f is a real function on X, the *$\mathfrak{J}$-essential supremum* of f is defined as

$$\sup_{\mathfrak{J}} f = \inf\{c \in \boldsymbol{R}: f^{-1}((c, +\infty)) \in \mathfrak{J}\}.$$

Two complex functions f and g on X are *equal $\mathfrak{J}$-almost everywhere* (in symbol, $f =_{\mathfrak{J}} g$) if and only if $\sup_{\mathfrak{J}} |f-g| = 0$.

Let $l_{\mathfrak{J}}^{\infty}(X)$ be the algebra of all complex functions f on X such that

$$\sup_{\mathfrak{J}} |f| < +\infty.$$

$\|f\|_{\mathfrak{J}} = \sup_{\mathfrak{J}} |f|$ is a pseudonorm on $l_{\mathfrak{J}}^{\infty}(X)$, $N_{\mathfrak{J}}(X) = \{f \in l_{\mathfrak{J}}^{\infty}(X): \|f\|_{\mathfrak{J}} = 0\}$ is an ideal of $l_{\mathfrak{J}}^{\infty}(X)$, and the quotient algebra $l_{\mathfrak{J}}^{\infty}(X) = l_{\mathfrak{J}}^{\infty}(X)/N_{\mathfrak{J}}(X)$ with the induced norm is, as one easily verifies, a Banach algebra with unit. The canonical image in $l_{\mathfrak{J}}^{\infty}(X)$ of a function $f \in l_{\mathfrak{J}}^{\infty}(X)$ will be denoted by $[f]_{\mathfrak{J}}$.

If A is a subset of X and f a function on X, then $f|_A$ stands for the restriction of f to A.

Let H be a discrete Abelian group. Given a function f on H and $x \in H$, $T_x f$ denotes the translate of f by x.

An ideal $\mathfrak{J}$ of subsets of H is *translation-invariant* if $A \in \mathfrak{J}$ implies $A + x \in \mathfrak{J}$ for all $x \in H$, where

$$A + x = \{y \in H: y = a + x, a \in A\}.$$

If $\mathfrak{J}$ is a translation-invariant ideal of subsets of H, $f \in l_{\mathfrak{J}}^{\infty}(H)$, and $x \in H$, then the element $T_x([f]_{\mathfrak{J}})$ in $l_{\mathfrak{J}}^{\infty}(H)$ is well-defined as the class $[T_x f]_{\mathfrak{J}}$, and we have

$$\|[f]_{\mathfrak{J}}\|_{\mathfrak{J}} = \|T_x([f]_{\mathfrak{J}})\|_{\mathfrak{J}}.$$

Given a linear continuous functional ξ on $l_{\mathfrak{J}}^{\infty}(H)$ and $x \in H$, $T_x \xi$ stands for the linear continuous functional on $l_{\mathfrak{J}}^{\infty}(H)$ defined by

$$T_x \xi([f]_{\mathfrak{J}}) = \xi(T_x([f])_{\mathfrak{J}}) \quad (f \in l_{\mathfrak{J}}^{\infty}(H));$$

we have, of course, $\|T_x \xi\| = \|\xi\|$.

For the remainder of the paper, we will assume that the group G is finite and the group Γ is discrete. The conditions for the action of G to be measurable will be now vacuous.

THEOREM 2.1. *Let $\mathfrak{J}$ be a translation-invariant ideal of subsets of Γ, Ω a subset of Γ, f a function in $l^{\infty}_{\mathfrak{J}}(\Gamma)$ such that $\|f\|_{\mathfrak{J}} \neq 0$ and for all $y \in \Omega$*

$$\sum_{g\in G} f(\cdot + gy) =_{\mathfrak{J}} f(\cdot) f(y). \tag{2.1}$$

Then there exists a complex bounded function F on Γ such that

$$\sum_{g\in G} F(x+gy) = F(x)F(y) \tag{2.2}$$

for all $x, y \in \Gamma$ and $F|_{\Omega} = f|_{\Omega}$.

Proof. Let ξ be a linear continuous functional on $l^{\infty}_{\mathfrak{J}}(\Gamma)$ such that $\xi([f]_{\mathfrak{J}}) \neq 0$. In view of (2.1), we have for all $y \in \Omega$ and all $z \in \Gamma$

$$\sum_{g\in G} T_z \xi([T_{gy} f]_{\mathfrak{J}}) = f(y)\, T_z \xi([f]_{\mathfrak{J}}),$$

whence

$$\sum_{g\in G} T_{z+gy} \xi([f]_{\mathfrak{J}}) = f(y)\, T_z \xi([f]_{\mathfrak{J}}).$$

Letting w denote the function $x \to T_x \xi([f]_{\mathfrak{J}})$, we have for all $y \in \Omega$

$$\sum_{g\in G} T_{gy} w = f(y)\, w.$$

By Theorem 1.2, there exists χ in $\hat{\Gamma}$ such that

$$f(y) = \sum_{g\in G} (gy, \chi)$$

for all $y \in \Omega$. To complete the proof, it suffices to take F to be the function

$$x \to \sum_{g\in G} (gx, \chi).$$

The function F in the statement of Theorem 2.1 is in general not unique. Below we present a theorem assuring the uniqueness of F in the case where $\Gamma \backslash \Omega$ is an element of a translation-invariant ideal of subsets of Γ.

THEOREM 2.2. *Let $\mathfrak{J}$ be a translation-invariant ideal of subsets of Γ, and F a complex bounded function on Γ satisfying* (2.2) *for all $x, y \in \Gamma$. Then F is the only complex bounded function in $[F]_{\mathfrak{J}}$ satisfying* (2.2) *for all $x, y \in \Gamma$. Moreover, $\|F\|_{\infty} = \|F\|_{\mathfrak{J}}$.*

Proof. Let $b\Gamma$ be the Bohr compactification of Γ, α the canonical homomorphism from Γ into $b\Gamma$, $AP(\Gamma)$ the algebra of all complex almost periodic functions on Γ, and $C(b\Gamma)$ the algebra of all complex continuous functions on $b\Gamma$. As is known, the mapping

$$C(b\Gamma) \ni h \to h \circ \alpha \in AP(\Gamma)$$

is a Banach algebra isomorphism.

Let ξ be a linear multiplicative functional on $l^\infty_{\mathfrak{J}}(\Gamma)$. The mapping $h \to \xi([h \circ \alpha]_{\mathfrak{J}})$ is a linear multiplicative functional on $C(b\Gamma)$ and as such it can be represented in the form $h \to h(\omega_\xi)$ for some $\omega_\xi \in b\Gamma$. If $\omega \in b\Gamma$ and $h \in C(b\Gamma)$, then

$$h(\omega) = \lim_{\alpha(x) \to \omega - \omega_\xi} T_x \xi([h \circ \alpha]_{\mathfrak{J}}), \tag{2.3}$$

passage to the limit being possible due to the fact that $\alpha(\Gamma)$ is dense in $b\Gamma$. It follows from the latter formula that every almost periodic function f on Γ is uniquely determined by its restriction to a subset of Γ whose complement belongs to $\mathfrak{J}$ and, moreover, $\|f\|_\infty = \|f\|_{\mathfrak{J}}$. The uniqueness of F and the identity $\|F\|_\infty = \|F\|_{\mathfrak{J}}$ result now from the fact that, by virtue of Theorem 1.1, any complex bounded function satisfying (2.2) for all $x, y \in \Gamma$ is a trigonometric polynomial on Γ.

3. Generalized Banach algebra valued solutions. Let A be a commutative semi-simple Banach algebra. Suppose Φ is a subset of the Gelfand space of A such that the only element of A whose Gelfand transform vanishes at all points of Φ is the zero element. Let $s(\Phi)$ be the algebra of all complex functions on Φ, equipped with the topology of pointwise convergence. When $\phi \in \Phi$ and $f \in s(\Phi)$, we write $\langle \phi, f \rangle$ for the value of f at ϕ. Let σ_Φ be the weak topology on A induced by Φ: σ_Φ has for a basis of neighbourhoods of the origin the sets

$$\{x \in A \colon \max_{1 \leq i \leq n} |\langle \phi_i, x \rangle| < \varepsilon\}$$

with $\{\phi_1, \ldots, \phi_n\}$ running over all finite subsets of Φ and ε running over all positive numbers. We identify A under σ_Φ with the topological subalgebra of $s(\Phi)$ consisting of all restrictions to Φ of the Gelfand transforms of elements of A.

An immediate consequence of Theorems 2.1 and 2.2 is the following

THEOREM 3.1. *For each $\phi \in \Phi$, let $\mathfrak{J}_\phi$ and $\mathfrak{J}'_\phi$ be translation-invariant ideals of subsets of Γ, and Ω_ϕ a subset of Γ such that $\Gamma \setminus \Omega_\phi \in \mathfrak{J}_\phi$. Let f be a function from Γ into A such that, given $\phi \in \Phi$,*

(i) *the function $x \to \langle \phi, f(x) \rangle$ is in $l^\infty_{\mathfrak{J}'_\phi}(\Gamma)$;*

(ii) *$\|\langle \phi, f(\cdot) \rangle\|_{\mathfrak{J}'_\phi} = 0$ implies $\|\langle \phi, f(\cdot) \rangle\|_{\mathfrak{J}_\phi} = 0$;*

(iii) *for every $y \in \Omega_\phi$*

$$\sum_{g \in G} \langle \phi, f(\cdot + gy) \rangle =_{\mathfrak{J}_\phi} \langle \phi, f(\cdot) \rangle \langle \phi, f(y) \rangle. \tag{3.1}$$

Then there exists a unique bounded function F from Γ into $s(\Phi)$ satisfying (2.2) *for all $x, y \in \Gamma$, such that, for each $\phi \in \Phi$,*

$$\langle \phi, F(\cdot) \rangle =_{\mathfrak{J}_\phi} \langle \phi, f(\cdot) \rangle.$$

Moreover, for every $\phi \in \Phi$,

$$\| \langle \phi, F(\cdot) \rangle \|_\infty = \| \langle \phi, f(\cdot) \rangle \|_{\mathfrak{J}_\phi}$$

and if $\| \langle \phi, f(\cdot) \rangle \|_{\mathfrak{J}_\phi} \neq 0$, then

$$\langle \phi, F(x) \rangle = \langle \phi, f(x) \rangle \quad \textit{for all } x \in \Omega_\phi.$$

Specializing the hypotheses of this theorem, we shall now derive a few results relating to the situation in which exact solutions of the equation under study take values in the same Banach algebra as initial generalized solutions.

THEOREM 3.2. *Suppose the unit ball of A is σ_Φ-compact. Let $\mathfrak{J}$ be a translation-invariant ideal of subsets of Γ. For each $\phi \in \Phi$, let Ω_ϕ be a subset of Γ with $\Gamma \backslash \Omega_\phi \in \mathfrak{J}$, and $\mathfrak{J}_\phi$ a translation-invariant ideal of subsets of Γ. Let f be a norm-bounded function from Γ into A such that, given $\phi \in \Phi$,*

(i) $\| \langle \phi, f(\cdot) \rangle \|_{\mathfrak{J}_\phi} = 0$ *implies* $\| \langle \phi, f(\cdot) \rangle \|_{\mathfrak{J}} = 0$;

(ii) (3.1) *holds for every $y \in \Omega_\phi$.*

Then there exists a unique bounded function F from Γ into (A, σ_Φ) satisfying (2.2) *for all $x, y \in \Gamma$, such that, for each $\phi \in \Phi$,*

$$\langle \phi, F(\cdot) \rangle =_{\mathfrak{J}} \langle \phi, f(\cdot) \rangle.$$

F is bounded in norm and, moreover, for every $\phi \in \Phi$,

$$\| \langle \phi, F(\cdot) \rangle \|_\infty = \| \langle \phi, f(\cdot) \rangle \|_{\mathfrak{J}}$$

and if $\| \langle \phi, f(\cdot) \rangle \|_{\mathfrak{J}_\phi} \neq 0$, then

$$\langle \phi, F(x) \rangle = \langle \phi, f(x) \rangle \quad \textit{for all } x \in \Omega_\phi.$$

Proof. In view of the preceding theorem, we need only to prove the existence part.

Let ξ be a linear multiplicative functional on $l^\infty(\Gamma)$ and ω_ξ the corresponding point in $b\Gamma$ such that (2.3) holds. Since, clearly, for each $\phi \in \Phi$ the function $x \to \langle \phi, f(x) \rangle$ belongs to $l^\infty_{\mathfrak{J}}(\Gamma)$ and is a trigonometric polynomial modulo $N_{\mathfrak{J}}(\Gamma)$, the right-hand side of the formula

$$(3.2) \qquad \langle \phi, F(x) \rangle = \lim_{\alpha(y) \to \alpha(x) - \omega_\xi} T_y \xi([\langle \phi, f(\cdot) \rangle]_{\mathfrak{J}}) \quad (x \in \Gamma)$$

makes sense. Reading the proofs to Theorems 2.1 and 2.2 we see that the assertion will follow once we prove that the function F defined by (3.2) is A-valued and is bounded in norm. By the compactness hypothesis, all reduces to proving that, given $x \in \Gamma$, $\phi \to \langle \phi, F(x) \rangle$, an element of $s(\Phi)$, lies in the closure of $f(\Gamma)$.

Suppose $a_1, \ldots, a_n \in C$ and $\phi_1, \ldots, \phi_n \in \Phi$ satisfy

$$\Big|\sum_{i=1}^{n} a_i \langle \phi_i, f(y) \rangle\Big| \leqslant 1$$

for all $y \in \Gamma$. Then, in view of (3.2),

$$\Big|\sum_{i=1}^{n} a_i \langle \phi_i, F(x) \rangle\Big| \leqslant 1$$

whatever $x \in \Gamma$. Every linear continuous functional on $s(\Phi)$ being of the form

$$h \to \sum_{i=1}^{n} a_i \langle \phi_i, h \rangle$$

for some $a_1, \ldots, a_n \in C$ and some $\phi_1, \ldots, \phi_n \in \Phi$, we reach the conclusion by utilizing the bipolar theorem.

In the sequel, if $g \in G$ and Σ is a subset of Γ, we write Σ_g for the set $\{x \in \Gamma \colon x - gx \in \Sigma\}$; for $x \in \Gamma$, we let

$$x - \Sigma = \{y \in \Gamma \colon y = x - s,\ s \in \Sigma\}.$$

THEOREM 3.3. *Let $\mathfrak{I}$ be a translation-invariant ideal of subsets of Γ, Ω a subset of Γ with $\Gamma \setminus \Omega \in \mathfrak{I}$, such that for each $x \in \Gamma$ the set*

$$\Omega(x) = (x - \Omega) \cap \Omega \cap \bigcap_{g \in G \setminus \{e\}} (\Omega - gx)_g$$

is non-void. For every $\phi \in \Phi$, let $\mathfrak{I}_\phi$ be a translation-invariant ideal of subsets of Γ. Let f be a function from Γ into A such that, given $\phi \in \Phi$,

(i) *the function $x \to \langle \phi, f(x) \rangle$ is in $l^{\infty}_{\mathfrak{I}_\phi}(\Gamma)$;*

(ii) *$\|\langle \phi, f(\cdot) \rangle\|_{\mathfrak{I}_\phi} = 0$ implies $\langle \phi, f(x) \rangle = 0$ for all $x \in \Omega$;*

(iii) *(3.1) holds for every $y \in \Omega$.*

Then there exists a unique bounded function F from Γ into (A, σ_Φ) such that (2.2) holds for all $x, y \in \Gamma$ and $F|_\Omega = f|_\Omega$.

Proof. Let F be the function from Γ into $s(\Phi)$ associated with f by the application of Theorem 3.1. In view of (ii), F coincides with f on Ω. Thus all that we need to prove is that F is A-valued.

Given $x \in \Gamma$, choose s in $\Omega(x)$, and let $t = x - s$. We have $s \in \Omega$, $t \in \Omega$, $s + gt \in \Omega$ for all $g \in G \setminus \{e\}$, whence $F(s) = f(s)$, $F(t) = f(t)$, and $F(s+gt) = f(s+gt)$ for all $g \in G \setminus \{e\}$. Consequently,

$$\begin{aligned} F(x) = F(s+t) &= F(s)F(t) - \sum_{g \in G \setminus \{e\}} F(s+gt) \\ &= f(s)f(t) - \sum_{g \in G \setminus \{e\}} f(s+gt), \end{aligned}$$

which yields the desired conclusion.

THEOREM 3.4. *Let $\mathfrak{I}$ be a translation-invariant ideal of subsets of Γ, Ω a subset of Γ such that $\Gamma\backslash\Omega\in\mathfrak{I}$. For each $\phi\in\Phi$, let $\mathfrak{I}_\phi$ be a translation-invariant ideal of subsets of Γ. Let f be a function from Γ into a subgroup B of the multiplicative group of all invertible elements of A such that, given $\phi\in\Phi$,*

(i) *the function $x\to\langle\phi, f(x)\rangle$ is in $l^\infty_{\mathfrak{I}_\phi}(\Gamma)$;*

(ii) $\|\langle\phi, f(\cdot)\rangle\|_{\mathfrak{I}_\phi}\neq 0$;

(iii) *for every $y\in\Omega$*

$$\langle\phi, f(\cdot+y)\rangle =_{\mathfrak{I}_\phi} \langle\phi, f(\cdot)\rangle\langle\phi, f(y)\rangle.$$

Then there exists a unique bounded function F from Γ into (A, σ_Φ) such that for all $x, y\in\Gamma$

$$F(x+y) = F(x)F(y)$$

and $F|_\Omega = f|_\Omega$. The range of F is contained in B.

Proof. Let F be the function from Γ into $s(\Phi)$ associated with f by the application of Theorem 3.1. Clearly, F coincides with f on Ω. We shall show that F is B-valued.

Given $x\in\Gamma$, let y be an element of $\Omega\cap(\Omega-x)$. Then $x+y\in\Omega$, and so

$$f(x+y) = F(x+y) = F(x)F(y) = F(x)f(y),$$

whence

$$F(x) = f(x+y)f(y)^{-1}.$$

The proof is complete.

The following is a generalization of de Bruijn's theorem mentioned in the Introduction.

THEOREM 3.5. *Let $\mathfrak{I}$ be a translation-invariant ideal of subsets of Γ, Ω a subset of Γ such that $\Gamma\backslash\Omega\in\mathfrak{I}$, and H a locally compact Abelian group. For each $\chi\in\hat{H}$ let $\mathfrak{I}_\chi$ be a translation-invariant ideal of subsets of Γ. Let f be a function from Γ into H such that for every $\chi\in\hat{H}$ and every $y\in\Omega$*

$$(f(\cdot+y), \chi) =_{\mathfrak{I}_\chi} (f(\cdot), \chi)(f(y), \chi).$$

Then there exists a unique function F from Γ into H such that for all $x, y\in\Gamma$

$$F(x+y) = F(x)+F(y)$$

and $F|_\Omega = f|_\Omega$.

Proof. The theorem follows immediately from the foregoing one upon identifying H with the group of all continuous characters of $\hat{H}$, which is a subgroup of the group of all invertible elements of the Banach algebra $l^\infty(\hat{H})$, and taking Φ to be the collection of all evaluation functionals on $l^\infty(\hat{H})$ corresponding to points in $\hat{H}$.

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
WARSZAWA

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